Mathematics as an Art form.

An essay about the stages of mathematics learning in an artistic evaluation of mathematical activity.

By

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Prologue

ART

You're in touch with the world that impinges on you,
Can you make any sense of this terrible brew?
You'd extract what is beautiful and what is good,
Or you gaze at the mess and just wish that you could!

If you have seized some good from this fast whirling brew,
You might then want to know if it's false or if true;
You'll transform what you've caught so that all understand,
Thus conveying your thoughts through the use of your hand.

This is where the artistic creation begins,
Creatures might have no legs or the fish have no fins,
What the brush puts on canvas has been through the fire,
"Let us get to the meaning!", we shall all inquire.

From the crucible of such creations' wild act
Comes a language that does not describe any fact!
Mental states are thus formed in recipients' minds
So that each one a meaning quite readily finds.

Zoltan P. Dienes, January 2004

Mathematics as an Art form.

1. Setting the stage.

Mathematicians often use aesthetic terms in describing the work they do. They talk about beautiful theorems, elegant proofs, cute short cuts, powerful symmetries, not to mention the architecture of the structure they are building. Many non -mathematicians are dumbfounded by such adjectives. They imagine mathematics as a dry, matter of fact series of strange statements, often expressed in some totally unintelligible language. How can anything like that be beautiful?

Of course beauty is in the eyes of the beholder so beauty arises out of the interaction between the beholder and that which is being beheld. It is open to question whether anything could be regarded as beautiful if there were no people who thought so. In the case of something created by Man, clearly the creator has to come into it, but one might doubt whether there was any meaning to calling a sunset beautiful in the absence of anyone thinking so!

A work of art is a CONSTRUCTION called into being by the artist who has somehow digested a certain number of inputs from the environment and has felt compelled to communicate the results to his fellow human beings.. A research mathematician plays with ideas, which he puts together into what may vaguely be called STRUCTURES. Then he messes about with these in cipient structures, trying to mould them, put them together, pull them apart in all sorts of ways until he hits on something that satisfies him. So the mathematician, in ways possibly quite similar to the ways of the artist, makes CONSTRUCTIONS. In this sense he is also an artist.

The problem for the educator is to find ways which would lead children to construct such abstract structures and encourage them to PLAY with them in the same way as they play with their toys. In order to PLAY you have to have a GAME. So our first question is

WHAT IS A MATHEMATICAL GAME?

In any normal game there is a starting point, then there are some rules that must be obeyed to move away from the starting point, and some criteria to decide when the game is finished. Games s uch as hockey, football, bridge, chess and so on all have such starting points, their own rules and some definite criteria which must be satisfied for the game to be finished. Some games are of the "patience" type, played by one person attempting to get to the end of the game as defined by the rules. Others are "win or lose" games where two people or two teams play against each other, and one side wins while the other side loses, unless it is possible by the rules to have a "draw".

Practically any mathemat ical structure will easily lend itself to the construction of either "patience" games or "win or lose" games. In fact, with a little ingenuity, one type can easily be turned into the other type. From the point of view of any aesthetic analysis, surely it is almost irrelevant which type of game we are talking about.

2. What is an artistic endeavour?

Let me try and face the knotty question of what Art really is. If we asked twenty artists, we should probably get twenty different answers to the question, so no body can claim the prerogative of "truth" in answering the question. So let me make a suggestion, which will hopefully be sufficient for what I wish to discuss in the sequel.

ART IS AN EXPRESSION OF AN URGE TO CREATE CONSTRUCTIONS WHICH EXTERNALIZE INTERNALIZED INPUTS FROM THE ENVIRONMENT.

If the above is the way in which we can view artistic endeavour, then in what sense is the mathematical game an Art form?

What have we taken in from the environment, which we have expressed as a construction?

To attempt to answer the question, let us look at how children behave when presented with material that they like to play with. When children are given "toys" with easily distinguishable attributes such as:

Colour, shape, size, thickness, number of holes, sex of person, attitude of person (sitting, standing, running etc.)

they will eventually reach a stage when they classify the toys having regard to one or more of the attributes in question. They might put all the boys on one side and all the girls on the other, or they might classify them also as sitting or standing, putting the sitting ones separately from the standing ones in the case of each sex. Putting their things "in the right places" involves classification. They will also put their classified piles in a certain order (boys before girls or the other way round etc.), taking into account one or more of the attributes that seem to them relevant. Somewhat later, especially if several children are taking part in the activity, different ideas emerge, so the classifications and orders already established are challenged and sometimes radically changed. It seems that, given suitably structured materials, children will feel the "urge"

TO CLASSIFY, then TO ORDER, then TO TRANSFORM.

It is hard to think of any part of mathematics that is not essentially a combination of the above three types of activity! The spontaneous reaction of children to structured materials leads us to a germinal form of the MATHEMATICAL GAME.

Children's primitive drawings, even scribbling, are the germinal versions of the beginnings of the urge to express and to represent using shapes and colours. It is a long journey from there to the drawing or painting of a work with artistic merit. It is likewise likely to be quite a trip from children's spontaneous play with material to the playing of sophisticated mathematical games.

3. The roles of the actors in an artistic activity.

Someone paints a picture, but someone else looks at it and receives the message. Someone writes the music, but s omeone else plays it and someone else again listens to the playing. It is the same in making up a mathematical game. Someone makes up the game but some else plays it. In fact someone, previously, must have invented the mathematics in question! All these are mathematical activities, although different mental processes are required for each type of activity. The inventor of the mathematics as well as the constructor of the game derived from that mathematics, would probably need to use his or her right hand brain quite a bit, whereas the "players", who have to work out the various strategies under the constraints of the game, are more likely to use their left hand brain.

In spontaneous play, namely in play without any prescribed rules, the player only has t o obey the constraints inherent in the material, such as those that relate to equilibrium and stability (tall towers tend to collapse more easily or it is not easy to stand up a very thin block etc.) or the geometrical properties of space (circles will not cover a surface, unless they overlap etc.), the number of objects available being a very obvious constraint!

An imperceptible passage takes place from the spontaneous stage towards the imposition of constraints not inherent in the material. Children migh t decide that two pieces of the same colour must not touch, or that in a row, the colours must come in a certain order. In this way the possibility of a goal to a "game" can evolve which will determine an endpoint to a game. This is the beginning of the stage of

INVENTING A GAME AND LEARNING TO PLAY IT

Once the game has evolved, with its rules and endpoint well specified, children will then KNOW the game and so can teach it to other children. This is the stage of

PLAYING THE GAME

Children are natural ly adventurous and more open to new things than we adults are! After having played their game a few times, as likely as not, they will want to change the rules, perhaps to correct a perceived difficulty or perhaps just to make the game more fun to play. This is the stage of

TRANSFORMING THE GAME INTO A NEW GAME

The new game naturally must be properly defined ("invented") and children will have to learn to play it, so the cycle starts again. Such a cycle might last an hour or two or maybe several months, depending on the game invented and on the children playing it.

Mathematics is a gold mine for an indefinite supply of games. Given any mathematical structure, a game can be invented whose constraints correspond exactly

to those present in the mathematical structure in question Some mathematicians would object, saying that the mathematics in question is already a game!

The problem for the educator is to harness the incredible energy released by what we describe as "play" in both artistically and mathematically meaningful ways. The conventional "lesson" clearly does not foot the bill. I recall my friend and colleague Robert Davies once saying that one way to stop little girls playing with dolls would be to introduce "doll playing lessons". What we need is not "lessons" but some of way of creating an environment involving the above described cycle of game-invention, game-playing and game-transforming, which would encourage children to express their natural urges

TO CLASSIFY, TO ORDER, and TO TRANSFORM

In the sequel I shall try to suggest some ways in which the educator might begin to solve this problem, illustrating the points by examples.

4. An example of inventing a game.

At a meeting In Visegrád, Hungary, of the International Study Group for Mathematics Learning, the Polish contingent of the delegates, led by Madame Krygowska and myself and helped out by Catalonia and Romania, set itself the task of trying to uncover the secrets of the construction of mathematical games. We took a pragmatic stand: we were going to observe ourselves as we did the inventing! The first thing was to locate a mathematical structure, which would act as the basis of the game or games. We all decided that a suitable structure, not too simple but not too complex, would be the eight element Galois field.

For the non-mathematical reader, here are the addition and the multiplication tables of the eight element Galois field (which you can safely ignore):

+	0	1	b	c	d	e	f	g
0	0	1	b	c	d	e	f	g
1	1	0	d	g	b	f	e	c
b	b	d	0	e	1	c	g	f
c	c	g	e	0	f	b	d	1
d	d	b	1	f	0	g	c	e
e	e	f	c	b	g	0	1	d
f	f	e	g	d	c	1	0	b
g	g	c	f	1	e	d	b	0

X	1	b	c	d	e	f	g
1	1	b	c	d	e	f	g
b	b	c	d	e	f	g	1
c	c	d	e	f	g	1	b
d	d	e	f	g	1	b	c
e	e	f	g	1	b	c	d
f	f	g	1	b	c	d	e
g	g	1	b	c	d	e	f

We can check for distributivity. For example (c + g) x d = 1 x d = d

But also (c x d) + (g x d) = f + c = d

So (c + g) x d = (c x d) + (g x d)

Of course this is only one trial, but I can assure the reader that the "trick" will work every time. This is one reason why the structure is called a field in mathematics.

It is obvious from the multiplication table that we are dealing with a cycle of seven. If we multiply an element by the same element seven times in succession, we shall return to the original element which we first multiplied. The addition table is less obvious. Of course our group was working under the advantage of "knowing" this structure, so we "knew" that the addition could be described as a "two by two by two" situation.

In order to create a physical environment which "embodied" the two by two by two idea, we chose three colours which we combined with each other in all possible ways to create the elements of our field "in concreto", So the elements of the game became eight paper plates, with the following sets of coloured objects placed in them:

Nothing, red, yellow, green, red & yellow, red & green, yellow & green, and finally red yellow & green.

These were to be our "toys". Now we had to formulate some rules, out of which a game could arise.

The addition rule was easy to "invent", Any two "platefuls" could be "added" by simply putting them together, but if one of the colours was present in both plates to be added, then this colour was left out of the "sum". This made sure that whenever you "added" two platefuls, you always obtained a plateful that was among the eight platefuls. For example

red & green + yellow & green = red & yellow

the green being left out since it is in both of the "addenda".

So we had an "adding rule" This is something, but in no way is it a game yet. We still had to decide how to start the game, what the rules were for playing it and how we would know that the game was over. Apart from that, we had to introduce the cycle of seven and in such a way that it should act as a multiplication, namely that it should be distributive over our addition. Last but certainly not least, we would need to inject some motivating factors to make sure that people would actually want to play our game!

As a bridge towards achieving the above aims, we thought up the following problem, to be solved by the "players":

Put seven empty plates out in a circular pattern. Then take four red counters, four green counters and four yellow counters and place a red counter in each one of four of the seven plates. Now "walk round the circle" several times, and note where the counters are in neighbouring plates and where you have to "jump" over a plate or plates to get to another counter. Now place the green counters, again one in each of four plates, following the same cyclic order as the one used for the red counters. The frequency of two green ones being next to each other or separated from each other, should occur in the same cyclic order as it does with the red ones. Having arranged the red and the green counters, do the same with the yellow counters. When you have finished, no two plates should contain the same combination of colours. If there are any "repeats", start with another way of placing the red counters and do the whole thing again.

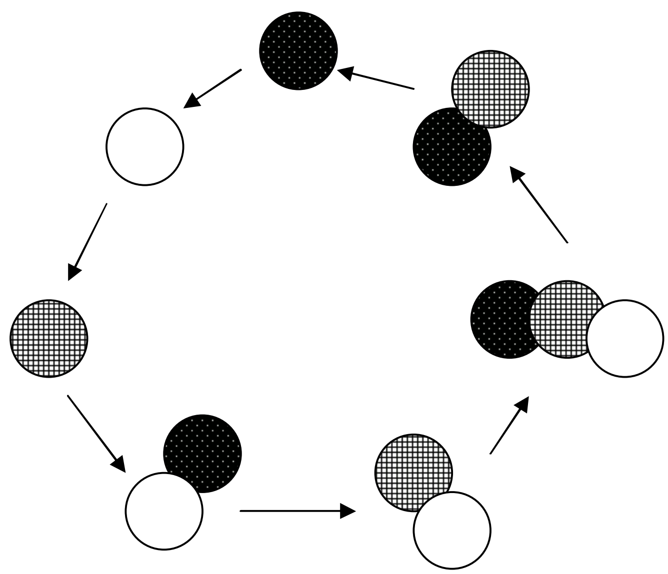
Of we course we all knew that there was only one way to solve the problem: there must be three consecutive "neighbouring plates" for each colour, the fourth one of each colour being separated from the "run of three" in each case by a "jump" over one plate or over two plates, depending on which way you were going round the circle..

This, strangely enough, is a necessary and sufficient condition for constructing the multiplicative group of our field, namely for constructing our seven cycle which must be su ch that any true additions should remain true if we moved both addenda and sum up or down by the same number of plates.

Now we have the bare bones of a game. But how do we play it?

Before facing that problem, let me draw a diagram to show an example of how the problem outlined above can be solved.

Following the arrows we see that of each colour there are three "in a row", followed by two "jumps" to get to the fourth one. It is also worth noting that the "sum" of any two neighbouring plates is obtained by jumping over the next plate in the direction of the arrows.



Having solved the above problem (in itself a fun activity as testified by those who have managed to solve it!), we could play a game in the following way:

STARTING SITUATION.
Player 1 arranges the seven plates in a circular pattern in any random manner

ALLOWED MOVES:
Player 2 exchanges the position of two of the plates according to his choice of plates. He continues doing this until the

FINAL POSITION
Which is a correct solution of the problem
The number of moves is counted. This number is player 2's score.

Players now change roles and player 2 gives player 1 the starting position. When player 1 reaches the final position, he or she counts his or her moves. The player with the lower score wins. If the scores are equal, the game is a draw.

There is strategy involved in choosing the starting position, players trying to make it that their opponents would need to use as many moves as possible to reach the solution position.
There is also st   rategy in selecting the pairs of plates whose positions must be exchanged in any particular move.

5. Analysis of the procedures described.

The steps appear to be the following:

(1) Examination and analysis of the chosen mathematical structure, in our case examination of the roles of the numbers 8 and 7, the former being the number of elements in the additive group, the second the number of elements in the multiplicative group.
(2) Finding a suitable concrete embodiment of the mathematical structure selected. In our case this was the set of all possible combinations of three colours.
(3) Verifying the suitability of the embodiment, by finding easily learned operations on the physical objects in question, which correspond to the mathematical operations in the selected mathematical structure.
(4) Finding a suitable problem "to play with". In our case this was the problem of placing the seven non-empty elements round a circular pattern so that the cyclic "rhythm" for each colour should be the same.
(5) Finding a way to "play with the problem". In our case this meant reaching a cyclic order of the plates in as few exchanges as possible, which solves the "equal rhythms" problem.

In the above, a somewhat sophisticated form of classifying comes into the activity, when a certain cyclic order is identified for one of the colours. The cyclic order of the other colours has to belong to the same class as the one to which the first ordering belongs.

Ordering is very obviously there in working with different cyclic orders.

Transforming is there in the "moves". The player transforms the plates from one cyclic order to another, until he or she hits on one that solves the problem A more radical form of transformation would be the posing of a similar problem for four colours or even five colours. Is there a particular cyclic order which will solve the four colour problem? In fact there is a necessary and sufficient condition for solving the four colour problem with fifteen plates. Colours must be arranged in the following cyclic order:

yes yes yes yes no no no yes no no yes yes no yes no

where yes means that a plate has that colour in it and no means that it does not.
In the case of five colours there are just two orders that will solve the problem.
To be honest, I have not worked out the problem for more than five colours. If you are mathematically inclined, have a go at the problem!

6. The role of further embodiments.

Although we can already see some of the "architecture" of the mathematical structure on which our game is based, it will be still better to "dress up " our structure in still different "clothes". We should then have a better appreciation of how the "addition" and the "multiplication" in the game are intertwined through the distributive principle. Playing about with different embodiments will enable us to see what is behind the physical embodiments, in fact we might begin to get a glimpse of the very bones of the mathematical structure in question. After all, mathematics is not really to do with plates in a circular pattern wi th colourd objects placed in them, just as the art of painting cannot be reduced to the paintbrush, can - vas and palette. Nevertheless the artist needs the paintbrush, canvas and palette in order to transmit his evolving construction to other fellow human beings. So how does the mathematician convey ideas to others? Unfortunately for the layman, mathematical language has grown to be so complex, so terse and non -redundant, that anything communicated in that medium is only accessible to other mathematicians, and even not to all of those! The alternative is to find a medium in which the layman can "play", just as the professional mathematician "plays", recording his "moves" using the appropriate symbol system. Quite possibly this medium could well be the

MATHEMATICAL GAME

which implies an

ART OF INTERPRETATION

The concert pianist makes gigantic efforts to interpret the amazing complexity of a Beethoven sonata to the lay public by decoding an abstruse symbol system and transforming it into an enjoyable artistic experience. Why could we not find ways to interpret the beauties of mathematics to the lay public through methods more accessible to the general public? Just as the pianist plays a role different from the composer, we could think of separating the

ACT OF DISCOVERY AND CONSTRUCTION

of the mathematician from the act of interpreting the construction to the public.

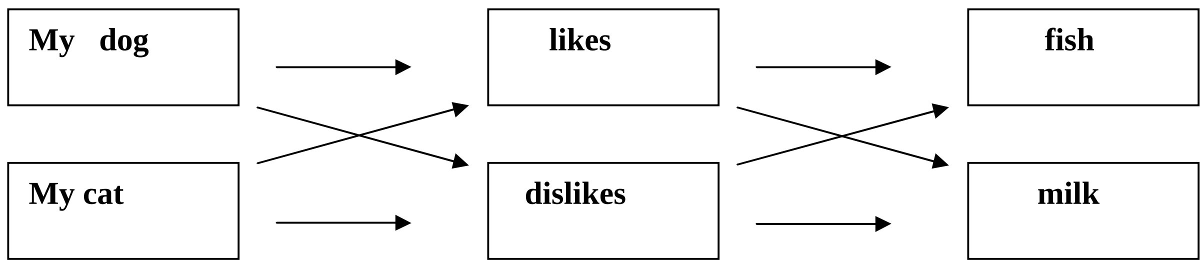
The search for other embodiments could be a part of the road towards developing techniques for the incipient Art of interpreting mathematics to the public.

I can suggest four different embodiments of the eight element Galois field. No doubt the reader can proceed to think of many more. Here they are:

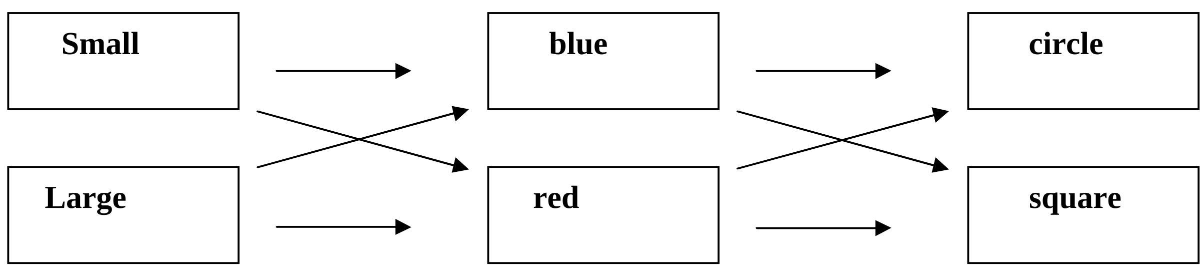
(a) The set of sentences we can construct out of two nouns, two verbs and two complements.
(b) The set of large and small, red and blue squares and circles out of a set of attribute blocks.
(c) The set of numbers { 1, 2, 3, 5, 6, 10, 15, 30 }
(d) The days of the week and a week's holiday.

The actual pieces for the above embodiments could be constructed out of schemata such as the following:

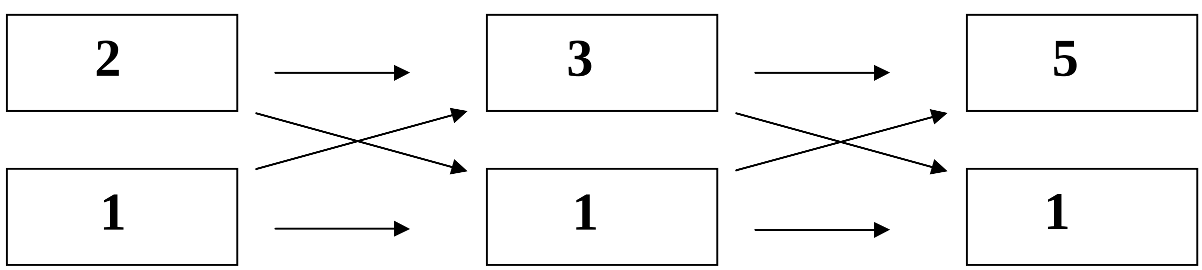
(a)



(b)



(c)



(where the arrows mean "multiply")

(d) Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday

The whole week

The "neutral element" in each case can be what arises out of the second line. So these neutral elements could be:

(a) My cat dislikes milk
(b) The large red square
(c) The number 1
(d) The whole week

In the case of the first three of the above embodiments it should be quite obvious how the game is played.

For the fourth embodiment the multiplication is just "moving up" a certain number of days in the week.. Sunday could be allotted the role of the multiplicative neutral, by saying "move up seven days ". "Multiplying" by Monday would mean "move up one day" "Multiplying" by Tuesday would mean "move up two days " and so on. For a "reasonable" addition system we could suggest the following cover story:

In a factory workers do a three day week. Two working days must be consecutive, then there must be a day of rest, followed again by a working day, after which there will be three days of rest. If anyone does two days' work in one, they are rewarded by a week's paid holiday. According to this system any two days determine the third day. For example:

Monday + Thursday = Tuesday

since if Monday and Thursday are working days, then Tuesday is the only day left which would make it that this worker works two consecutive working days, followed by a day of rest, followed by a third working day.

Having played at "inventing embodiments" , we now have two jobs ahead of us:

(i) To ascertain which of the embodiments fits our mathematical structure best, bringing out its architecture in the most visible and pleasing form;
(ii) To find the best bridges to pass from one embodiment to another.

Possibly the most visual embodiment is the one using attribute blocks. The colours, shapes and sizes of the blocks stand out very vividly and the "rhythm" can be checked and enjoyed by simply casting a glance at the various attempts to make the three "rhythms" coincide.

The numerical embodiment helps us to become more aware of how the elements of the game are constructed by multiplying prime factors

The linguistic embodiment makes us aware of the different roles played by the subject, the predicate and the complement in a statement.

The days of the week embodiment concentrates on the "multiplicative aspect" of our "mathematical toy". The very well internalized sequence of the days of the week brings out the "seven-ness" of the multiplicative part of the structure. The "additive

aspect" is not so clear, as in order to "add" two days, we have to go through a process of reasoning, the "sum" is not immediately obvious.
The easiest way to pass from one embodiment to another is to associate the building blocks of one embodiment to the building blocks of the other embodiment. For example we could make the following bridge:

Small red square → My dog dislikes milk → 2

Large blue square → My cat likes milk → 3

Large red circle → My cat dislikes fish → 5

The elements from the first three embodiments being those that differ in just one way from the neutral, so the other elements can be "built up" by addition, starting from these three elements. It is a bit harder to build a "sensible" bridge to connect with the days of the week. That challenge is left to the reader.

For the sake of completeness, here is the rest of the suggested "bridge":

Large red square → My cat dislikes milk → 1

Large blue circle → My cat likes fish → 15

Small red circle → My dog dislikes fish → 10

Small blue square → My dog likes milk → 6

Small blue circle → My dog likes fish → 30

7. How to bridge an embodiment to itself!

We have built bridges between embodiments, so we can pass easily from one concrete version of our mathematics to another. But what about building a bridge from an embodiment back to itself, in a kind of "loop"? Each element will be transformed into some element of the same embodiment, maybe even into the very same element. But it might be wise to make a proviso, namely that if for three elements A, B and C it is true that

Element A + Element B = Elemenent C
then it should also be true that
Transformed element A + Transformed element B = Transformed element C

Such a transformation of a structure "into itself" is known in mathematics as a

HOMOMORPHISM

So we are looking for homomorphisms in the eight element group, represented concretely by one or the other of our embodiments.

Clearly a multiplication would provide such an homomorphism. Take for example the cycle:

My dog likes milk 1	My cat dislikes fish b	My dog dislikes milk c	My dog likes fish d	My dog dislikes fish e	My cat likes fish f	My cat likes milk g

Taking it from left to right, the last sentence being followed again by the first one in the row. The sentence marked 1 could be the multiplicative neutral, the sentence b would move up one notch, the sentence c would move up two notches and so on.

Since the additive neutral is "My cat dislikes milk", the parts of the neutral sentence could be denoted by zeros, so my cat = 0, dislikes = 0 and milk = 0. Therefore we would have my dog = 1, likes = 1 and fish = 1 and we would use binary addition, the sequence now looking like this:

110 → 001 → 100 → 111 → 101 → 011 → 010 → 110

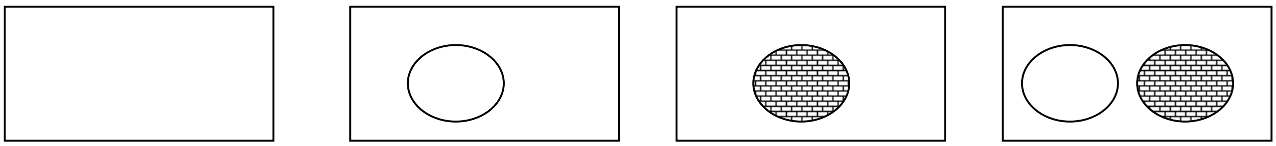
We can easily check that b + c = e and if multiplying by b means "move up one notch" then

(b x b) + (c x b) = c + d = (e x b) = f

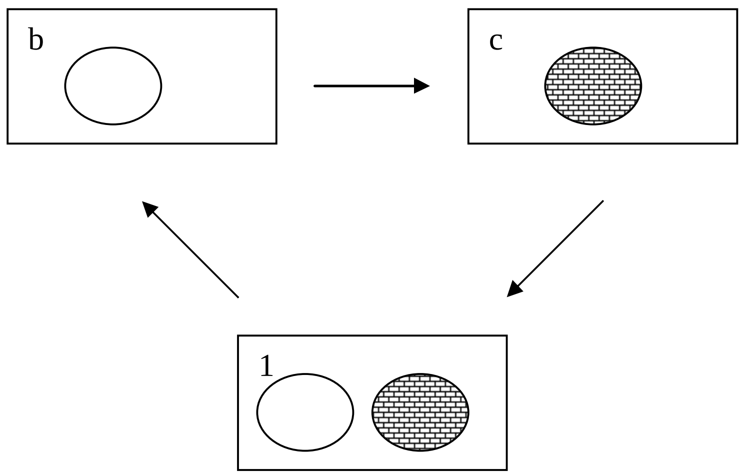
so "moving up one notch" turns a true addition into another true addition, so we have a homomorphism. Naturally, this can be done with all the "multipliers". So there are at least as many homomorphisms as there are elements in the game, namely seven, but probably there area whole lot more!

Could this bridging the game into itself provide the elements of yet another game, "born" from the one we have just made up?

I can illustrate this possibility by looking at a simplified version of our game, namely by reducing the number of colours to two. So our bricks out of which the simplified game is built are:



Adding is defined as before, by putting elements together but leaving out anything that is common to the two addenda. Multiplication can be done through moving up in the simplified cycle:

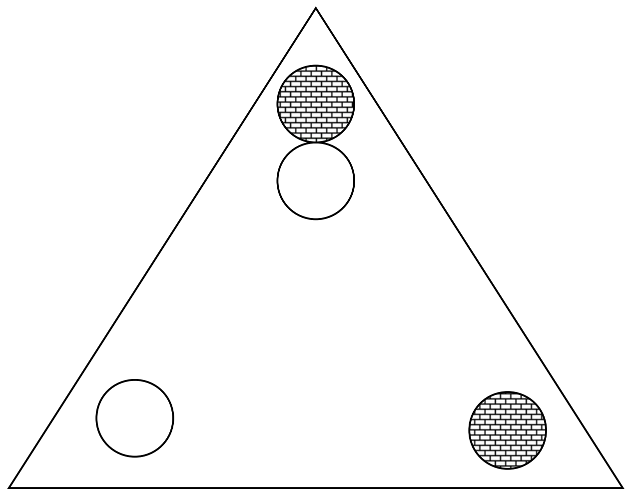


Multiplying by 1 would mean "stay where you are", multiplying by b would mean move up one notch, by c move up two notches (or back one notch).

The above three multiplications give us straight away three homomorphisms. But there are three more, which are

(i) b → c → b, (ii) b → 1 → b, (iii) c → 1 → c

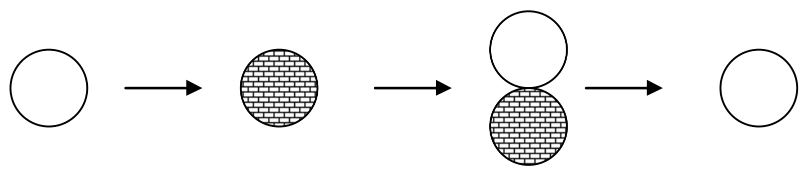
Strangely enough, we can realize these changes by referring to the rotations and reflections of the equilateral triangle. All we have to do is to draw these figures on both sides of an equilateral triangle as suggested below:



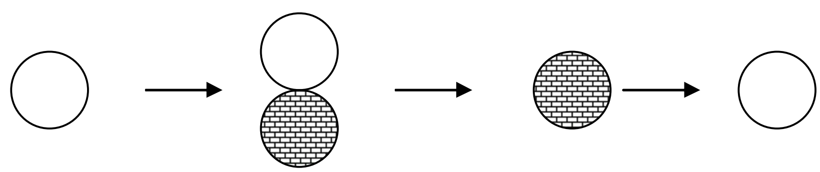
There are three rotations:

(i) leave the triangle as it is

(ii) rotate so you get the changes



(iii) rotate so you get the changes



and there are three reflections, namely

(iii) reflect about the axis through the white circle

(iv) reflect about the axis through the bricked circle

(v) reflect about the axis through the two circles.

Note that each rotation has to be done clockwise when one face is up but counterclockwise when the other face is up (let us not forget that a transparent clock would move in th e counter clockwise sense if we looked at it from behind, but for the clock it is just the same!)

We have managed to put together three structures, which have been intertwined with each other, one of them being a game about a game (a meta-game?). These are

(i) the two by two system, represented by two patterns or colours, each one being either absent or present, giving us an "addition”

(ii) the three cycle, used as a "multiplication" as a "moving up the cycle", distributive over the addition,
(iii) the rotations and reflections of the equilateral triangle, obtained through the scandalous idea of mirroring the game back into itself!

The appreciation of the architecture of such a combination of structures is surely akin to the appreciation of the impression conveyed by a painting by Picasso or a concerto by Mozart.

The fact that our two by two game can be mirrored into itself in six different ways surely shows that there is some inherent symmetry about the structure on which the game is based. If we tried to make a "game about the game", starting with a cycle of four, instead of a two by two situation, we would only find two ways of mirroring such a cycle into itself. It seems that such a cycle is less inherently symmetrical than the concept of a two by two.

Children tend to think symmetrically. When they draw a house, they will usually put the door in the middle and have a window on either side of it. If they put a tree on one side, they will usually put a tree on the opposite side as well. In some cognitive experiments I did with Malcolm Jeeves in Adelaide we taught children the four cycle, and another group learned the two by two group. In the predictions they were asked to make, many of the four cycle learners made predictions as though they were learning the two by two group, in spite of the fact that they had had no evidence to that effect. Maybe the internal symmetry of the two by two group was more compelling than the physical evidence they had had for the cycle of four. When we elicited questions about this after the tests, many children would say "Well, that's how I thought it would go!". The first step is symmetry and departure from symmetry the second step. We first have to learn how to draw eyes that look like real eyes, using the symmetry of the human face, before we can use the asymmetry of the position of the eyeballs to convey a particular expression! It seems likely that we are in a similar situation in learning structures with different degrees of inherent symmetries.