Circular Villages.

(Some fun with the associative rule or “bunching rule”).

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1. The picnic arrangements.

There is a village in Ruritania on one of the outlying islands where seven houses have been built in a circular pattern. There is also a house for the policeman, which has been built right in the middle of the circle. The number of the policeman’s house is 0, the other houses have numbers 1, 2, 3, 4, 5, 6 and 7, arranged in the clockwise sense.

The inhabitants decide to make a rule about getting together from different houses for having picnics. They decide that sometimes they will have a big picnic, and sometimes a small one, but most of the time a small one. They also decide that at each picnic people from three different houses must participate, except when the policeman’s family is involved, in which case only one other house will be involved, but this house will then bring a big picnic. The three houses must be so chosen that two of them are next to each other, and the third one is the next house but one (counted clockwise) from the two that are together.

They use an octahedron as a die for deciding which houses must go on the next picnic. The numbers 0, 1, 2, 3, 4, 5, 6 and 7 are written on the eight faces of the octahedron. They throw the die twice, which decides which two out of the three houses will take part in the next picnic. If they throw the same number twice, this number house must bring a large picnic and the other house included will be the policeman’s house. If a 0 and another non-zero number is thrown, then this non-zero house will be the “third house” by increasing the size of the picnic to a large one. In the exceptional case when two zeros are thrown, then the policeman’s family are the only ones to go to the picnic and they have really sumptuous one!.

2. The addition rules of the addresses in the village.

In the above way, any two numbers out of 0, 1, 2, 3, 4, 5, 6 and 7 will determine a third number. Try to practice giving yourself any two of these eight numbers, and see if you can always work out the third number! We shall say that the "sum" of any two addresses in a picnic party is the address of the third house in that party.

Here is the table, for reference:

plus	1 +	2 +	3 +	4 +	5 +	6 +	7 +
1 =	**0**	**4**	**7**	**2**	**6**	**5**	**3**
2 =	**4**	**0**	**5**	**1**	**3**	**7**	**6**
3 =	**7**	**5**	**0**	**6**	**2**	**4**	**1**
4 =	**2**	**1**	**6**	**0**	**7**	**3**	**5**
5 =	**6**	**3**	**2**	**7**	**0**	**1**	**4**
6 =	**5**	**7**	**4**	**3**	**1**	**0**	**2**
7 =	**3**	**6**	**1**	**5**	**4**	**2**	**0**

3. Is this "addition" associative?

Now choose any three addresses (numbers), preferably three that do not form a picnic party. Is it true that

The sum of the first two addresses
added to the third one,
gives the same address as the first number
added to the sum of the last two ?

Let me try three addresses: take for example 2, 3 and 6.

We know that 2 + 3 = 5 and that 5 + 6 = 1, since (2, 3, 5) and (5, 6, 1) are houses giving permitted picnic parties, so we get 1 the first way.

But 3 + 6 = 4, since (3, 4, 6) is a picnic party, and 2 + 4 = 1, since (1, 2, 4) give us a picnic party, so we get 1 working the additions out "bunched" in the second way.

We say that we "associate" when we "bunch". So if we get the same sum whichever way we associate, or bunch our numbers, we say that the addition is **associative.**

The addition I have proposed is in fact associative, although I have not proved it to you, as I should have shown that it worked for ANY three numbers!

4. Enlarging the circle.

A new family arrives, and they build them a house, whose address will be number 8. Naturally house number 8 will be between house number 7 and house number 1.

How do the rules have to be changed now that we have one more house?

Why not try the following variation? Picnics can be arranged in the following ways:

Either

(i) the three houses must be in this order (chosen houses being marked YES)

YES NO YES YES in the clockwise sense

or

(ii) two of the houses must be on opposite sides of the circle, the third one being the policeman's house.

or

(iii) A family from just one house go for the picnic, a very sumptuous one! (this could be the policeman's family, or any other). This counts as three . houses!

You will find that having chosen any two households, one and only one of the above rules will apply and it will be possible to find the third household that will take part in the picnic.

What about the bunching problem we tried with seven houses round the circle? Would it still work with eight houses?

Let us try three houses, for example the houses 3, 4 and 5.
We know that 3 + 4 = 1, since (1, 3, 4) is a correct picnic set by rule (i).
But 1 + 5 = 0 since (0, 1, 5) is a correct picnic set by rule (ii).
So adding the third address to the sum of the first two, we get the policeman's house.
Now 4 + 5 = 2, since (2, 4, 5) is a correct picnic set by rule (i).
But 3 + 2 = 8, since (8, 2, 3) is a correct picnic set again by rule (i)
So adding the first address to the sum of the last two, we get 8, which is not 0.
So the "bunching rule" breaks down in this case! This addition is NOT associative!

5. Changing the rules for making the addition associative.

Is there any way we can "save" it?
How about trying the following "variation" of the rules so far proposed:

The sum of any two addresses could be the address
OPPOSITE the third address determined by the two chosen addresses.
So, for example 3 + 4 will no longer be 1, but 5.
Let me go through the previous calculation, but with the "amended" rules:

We know that 3 + 4 = 5, and 5 + 5 = 1 since (5, 5, 5) = sumptuous picnic for 5!
Then 4 + 5 = 6 since (2, 4, 5) give us a good picnic set,
and since the house 2 is the one opposite the house 6.
So we must now find 3 + 6, , which is 1, since (3, 5, 6) form a good picnic set, and
house 1 is opposite house 5.
So we get house 1 whether we add the third house to the sum of the first two
or whether we add the first one to the sum of the last two!

Have we saved the "bunching rule"?

At this point you should try a number of ways of picking three houses, and bunch them both ways:

(i) by adding the sum of the first two to the third,
(i) By adding the first to the sum of the last two.

If you get the same final house, whichever way you "bunch" the adding, you have saved the rule. If it does not work every time, we have not saved it. Try and decide for yourself!

6. Making a larger circle out of two circles.

On this remote island there is another village with seven houses in a circle, with the policeman's house in the middle of the circle. They have asked the people in the expanded village, the one with eight houses and one in the middle, if they could join them and make one large circle. At first they did not know how to handle the two policemen, but finally they decided that a larger house could be built in the middle of the larger circle, and the fifteen houses would be built round this new, larger policeman's house and the two policemen would share duties.

They still wanted to preserve their picnic ideas, so that three households would take part in any picnic. They found that this could be done by keeping to the following rules for choosing three houses (a large picnic is like counting the same house twice):.

(i) Between any two of the three houses taking part in a picnic there must be four that do not take part
So the pattern must look like this:

YES NO NO NO NO YES NO NO NO NO YES

(ii) Two of the houses taking part in a picnic must be right next to each other, and there must two non-participating houses between the second and the third houses of the picnic party. So the pattern is like this:

YES YES NO NO YES

(iii) Two of the houses taking part must have one house not taking part between them and there must be five houses not taking part between the second and the third houses whose inhabitants do take part in the picnic.
So the pattern must look like this:

YES NO YES NO NO NO NO NO YES.

(iv) One household brings a large picnic and joins with the policeman's family.

Is it true, for the above rules, that any two addresses that might be chosen, can always only have one third address for joining in the picnic? In other words, is the composition of the picnic party already decided by the choice of two households who will take part?

Try a number of ways of choosing two addresses and see if you can always work out what the third address must be.

The other thing we should look at is the "bunching rule". We could choose any three addresses and see how the adding in the two different ways works out. Let me take three addresses at random, like say the addresses (3, 5, 7)

Let as add (3 + 5) to 7 using the new rules.. We know that 3 + 5 = 11 by rule (iii) since according to this rule the houses (3, 5, 11) form an allowed picnic set.

Then 11 + 7 = 8 by rule (ii) as according to this rule (7, 8, 11) form an allowed picnic set of houses.. So (3 + 5) + 7 = 8.

Now let us find 3 + (5 + 7). We know that 5 + 7 = 13 by rule (iii). But we also know that 3 + 13 = 8, by rule (i), since (3, 8, 13) form an allowed picnic set by that rule.

So it seems that we get 8, whichever way we bunch the adding!

Of course we might have just hit on a lucky example. You should now try a few more and see if the bunching works out every time. If it always works out to be the same sum, whichever way we bunch the adding, then we have "saved the rule". Try to decide for yourself. Is this addition associative?

We should really add the rule that the policeman's family can go out all by themselves and have a sumptuous picnic. That would count the policeman's address three times.

We have made circular villages with circles of 7, of 8 and of 15. Perhaps it is not altogether a chance that:

7 = (2 x 2 x 2) - 1, 8 = (3 x 3) - 1 and 15 = (2 x 2 x 2 x 2) - 1

If you can see what I am aiming at, you might try to construct circular villages with circles of 24, of 26, of 31, of 48 and so on. If you want some hints you can e-mail me at zoltan@zoltandienes.com

P.S. Just for fun, let me give away some rules for the 24 circle that will work. In the following table the first number is the number of uninvited households between the first and the second invited households, the second being counted clockwise from the first. The second number tells you how many houses on from the second house , or how many houses back from the first house you will find the third household invited to take part in the picnic. If the invited households have 7 uninvited households between them, then the third one is the one half way between the first and the second houses. In all other cases it is either "on from the second" or "back from the first", "on" meaning clockwise and "back" meaning counter clockwise.

Uninvited households in between	Third invited household
0	2 on
1	7 on
2	7 back
3	1 on
4	9 back
5	6 on
6	1 back
7	half way between the two
8	2 back
9	3 back
10	5 back
11	policeman's household

We might as well try just one "bunching test", not that it "proves anything", since the "bunching" must work in all cases! But let us try

$$(1 + 5) + 7 \quad \text{as opposed to} \quad 1 + (5 + 7)$$

Between 1 and 5 there is a "gap" of 3, so we go 1 house "on" from house 5, and we get house 6.. But $6 + 7 = 9$, since there is a "zero gap" and so we must move "on" to the second house from house 7, namely to house 9. So $(1 + 5) + 7 = 9$.

Now let us do $1 + (5 + 7)$, we see that there is just one house between houses 5 and 7, so we move 7 houses on from house 7, which gives us house 14. So $(5 + 7) = 14$
Now what is $1 + 14$? The gap between 14 and 1 is 10, so we move 5 houses back from House 14 and we get h House no.9 again!

I can assure you that this is not a chance event! If you try any other set of three houses, "adding" their numbers bunched either one way or the other way, you will get the same "sum". You may be wondering how on earth I dug up such a set of rules. I can give you a hint: it is to do with "numbers" mod 5. If you want to know more, just e-mail me!

The secrets of the circular villages.

1. Adding the days of the week using fruit salads.

My children used to like making the fruit salad to have at the end of a meal. They used to put apples, pears and grapes in the fruit salad. But they wanted variety, so sometimes they just used one of these ingredients, sometimes two and sometimes all three. So they had seven ways of making the fruit salad, one for each day of the week.

This is how they arranged the fruit salads:

Monday	apples and grapes
Tuesday	apples
Wednesday	pears
Thursday	grapes
Friday	apples and pears
Saturday	pears and grapes
Sunday	apples pears and grapes

They found that if they did this, on any four consecutive days, on the first, second and fourth days they had apples twice or not at all, pears twice or not at all and grapes twice or not at all. They also found that they had the same fruit in the fruit salad on three consecutive days, then they did not have it on the fourth day, but they had it again on the fifth day. They also found that they used each kind of fruit four times during the week. They were very pleased with this arrangement.

One day I found them playing an "adding game" with the days of the week. When I asked them what they were doing, they explained it to me like this:

Take any two days of the week. To "add" these days, all you have to find is the third day such that on that day, together with the two chosen days to be "added", every kind of fruit should have been used twice or not at all, counting all the three days.

"Well", I went on "What is Monday + Wednesday?"

"It's Sunday!", answered the children all at once, "since we would have eaten apples twice, pears twice and grapes twice on Sunday, Monday and Wednesday!"

"What do you get if you add a day to itself?, I ventured to ask them.

“There would be two or none of each fruit in the sum already, so we cannot put anything in the salad for the sum-day!”, objected one of the children.

“We can make a pineapple salad and have it as a treat before going to bed”, suggested another child. “So a day added to itself is equal to a bedtime treat!

2. Is this adding associative?

“Do you think your addition is associative?”, I asked them.

There was a blank stare of incomprehension.

“What’s that?”, they asked me.

“Take any three days so that the sum of two of them is not equal to the third”, I said.

“All right”, said the children, “ what about Monday, Tuesday and Wednesday?”

“All right”, I replied, “Now add Monday to Tuesday and then add the sum to Wednesday. Then add Monday to the sum of Tuesday and Wednesday. Do you get the same day in both cases?”

The children were quick to work this out. This is how they did it:

Monday + Tuesday = Thursday, and Thursday + Wednesday = Saturday.
But Tuesday + Wednesday = Friday, but Monday + Friday = Saturday.
So the answer is Saturday both ways.

I explained that if you do this and every time you get the same sum both ways, the addition is associative. You “associate” Monday with Tuesday and then add the result to Wednesday, and then check whether Monday, added to Tuesday and Wednesday “associated together” gives the same answer. But for the addition to be associative, it has to work for any three days!

The children took a long time adding their days, but they could not come up with an example when it did not work out. But they were not sure if their addition was associative, because, they said, they had not tried all the possible sets of three days, and there were too many of them to try!

3. The addition table for the days of the week.

+	**Monday**	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Saturday**	**Sunday.**
Mon	**Pineapple**	**Thurs**	**Sun**	**Tues**	**Satur**	**Fri**	**Wednes**
Tues	**Thurs**	**Pineapple**	**Fri**	**Mon**	**Wednes**	**Sun**	**Satur**
Wednesday	**Sun**	**Fri**	**Pineapple**	**Satur**	**Tues**	**Thurs**	**Monday**
Thursday	**Tues**	**Mon**	**Satur**	**Pineapple**	**Sun**	**Wednes**	**Fri**
Friday	**Satur**	**Wenes**	**Tues**	**Sun**	**Pineapple**	**Mon**	**Thurs**
Saturday	**Fri**	**Sun**	**Thurs**	**Wednes**	**Mon**	**Pineapple**	**Tues**
Sunday	**Wednes**	**Satur**	**Mon**	**Fri**	**Thurs**	**Tues**	**Pineapple**

4. Adding days in an eight-day cycle.

On another day I found that they had made up a game about making the vegetable salad. They were fond of potatoes and carrots, so they decided that on different days they would put a different number of potatoes and carrots into the salad. They never put more than two potatoes, nor more than two carrots into the salads. They found that to have all the possible ways of making the salad, they needed eight days, not seven. So they made up the following eight day cycle:

Day 1	**1 potato**
Day 2	**1 carrot**
Day 3	**1 potato & 1 carrot**
Day 4	**1 potato & 2 carrots**
Day 5	**2 potatoes**
Day 6	**2 carrots**
Day 7	**2 potatoes & 2 carrots**
Day 8	**2 potatoes & 1 carrot**

They told me that they arranged the sequence in such a way that during any four consecutive days, on the first, third and fourth days they will have used either 3 or 6 potatoes altogether, as well as either 3 or 6 carrots altogether. Then they told us that to "add" any two of the above eight days, all you had to do was to find the third day so that on all the three days together 3 or 6 potatoes and 3 or 6 carrots would have been used during the three days.

"What happens if you wanted to add Day 4 and Day 8 ?", I asked them.

They were stumped, because on those two days the 3 or 6 potato and carrot rule was already fulfilled.

"Well", replied one of the children, "in that case we shall have a party on any chosen day, in the course of which there will be a pineapple salad served with no potatoes and with no carrots!"

"So the adding game really has nine elements in it, the eight days and the pineapple salad party, right?", I concluded.

In that case, for example, Day 1 + Party = Day 5, since Day 1 and Day 5 already fulfill the rule, so the Pineapple Salad Party is the third element, it was finally concluded.

5. Is this addition associative?

“Is this addition associative”, I asked.

“Let’s try and see whether it is”, suggested the children.

So they tried Day1, Day 2, and Day 3.
Clearly Day 1 + Day 2 = Day 7 and also
Day 7 + Day 3 = Party..
But Day 2 + Day 3 = Day 8,
and Day 1 + Day 8 = Day 6.
So (Day 1 + Day 2) + Day 3 = Party
but Day 1 + (Day 2 + Day 3) = Day 6.

It is clear that this “addition” is not associative.

“Do you have a rule for adding a day to itself?”, I asked.

“I think our rules will work like this. Adding a salad to itself, we shall have either 2 or 4 of each vegetable. To get 3 or 6 of each, we just have to add the same salad again!”, suggested one of the children. “But I don’t like the idea of our addition not being associative”, this child added.

6. Can changing the rules somewhat make it associative?

“Is there a way of changing the rules so that it becomes associative?”, asked another one of the children.

“Actually, I do believe there is”, I replied,” Instead of taking the sum as you have thought it up, why not calculate that sum and then move the sum up four days in your cycle?”

“It seems a weird thing to do, but let’s try”, agreed the children.

With Day 1, Day 2 and Day 3 this is how it works out:

Day 1 + Day 2 = Day 3, and Day 3 + Day 3 = Day 7

“We get Day 7, because with the old rules Day 3 + Day 3 would have been Day 3, so taking the day four days later, we get Day 7”, the children tried to explain to me.

“But Day 2 + Day 3 would have been Day 8 by the old rules, which makes it Day 4 with the new rules, going on four days”, reasoned another one of the children, “so we have to find Day 1 + Day 8.. By the old rules it would have been Day 3, so adding four days we get Day 7. So we get Day 7 both ways!”, concluded this child.

"It seems we might have saved your addition from falling into the pit of being non-associative!", I suggested, "but you have to try more sets of three days, so you can be more certain!", I added.

They tried lots of times, and every time the two ways of computing the "sum" turned out to be the same day of their cycle! Of course sometimes they got the Pineapple salad, but even this did not seem to spoil things, since if they ended up with the Pineapple salad one way, they ended up with it the other way as well!

7.More fruit makes longer fruit salad cycles.

The children liked the pineapple so much in the fruit salad that they wanted to incorporate it into their regular salad-sequence. In order to get all the possible variations of the different fruits, they found that they needed a cycle of 15 days, so they made up the following sequence:

Day 1	apples	Day 9	apples & grapes
Day 2	pears	Day 10	pears & pineapple
Day 3	grapes	Day 11	apples pears & grapes
Day 4	pineapples	Day 12	pears grapes & pineapple
Day 5	apples & pears	Day 13	All four fruits
Day 6	pears and grapes	Day 14	apples grapes & pineapple
Day 7	grapes & pineapple	Day 15	apples & pineapple
Day 8	apples, pears & pineapple		

"We have each fruit eight times in the 15 day cycle", they declared, " also on the first, second and fifth of any five successive days we have every fruit twice or not at all.. This fact will be good for finding an adding rule for our days", they continued to tell me.

In fact they did decided that for any three days for which it was either two or zero times that they used any of the fruits would be the test. For any such three days any two of them would "add up" to the third one. A day "added to itself" would have no apples, no pears, no grapes and no pineapples, so they thought that the sum of any day added to itself would be a party at which the fruit salad was made entirely of passion fruit.

I asked them how they could tell whether any given set of three days out of their cycle of 15 would be such that any two of them would "add up" to the third one. They replied like this

"Either two of the days would need to be together, then after a gap of two days would come the third day,

or the three days must be evenly spaced along the cycle,

or you could have two days separated by one, the third one being the sixth day after the second day of the two."

“But what about the passion fruit salad?”, I asked

“If there was passion fruit and you added it to one particular day, then the sum would be that same day. If you added a day to itself, you would get the passion fruit salad”, they replied logically.

“Which of these three possible rules will you use if you want to add Day 3 and Day 9?”, I asked.

“That would be Day 1, and it is the third rule, because between Day 1 and Day 3 there is one day, and Day 9 is 6 days after Day 3, as required by that rule”, they replied. “And you can see that we get apples twice and grapes twice on those days, and we do not get any pears and do not get any pineapples!”, they went on explaining.

“What about adding Day 4 to Day 9?”, I asked.

“That will be the even spacing rule”, they assured me, because Day 9 is 5 days after Day 4, so another 5 days after that you get Day 14, and another 5 days after that you get Day 4 again!”, they said, “We get no pears, and out of the other three fruits, we get two servings from each during those three days!”

So they could now do their adding without thinking about the fruit in the fruit salads, they merely had to look for the appropriate rule to use, to find the “sum” of any two days.

8. Is this addition associative?

They also tested to see whether this addition was “associative” and they found no sets of three days for which the two ways of adding came to different answers. So they were fairly convinced that their addition rules did in fact lead to an associative addition. What luck!.

Here is an example:
Let us see what happens to Days 1, 2 and 3.
Day 1 + Day 2 = Day 5 by rule 1 (Days 1 and 2 being together),
and Day 3 + Day 5 = Day 11 by rule 3 (Days 3 and 5 having a day in between).
Day 2 + Day 3 = Day 6 by rule 1 (Days 2 and 3 being together)
and Day 1 + Day 6 = Day 11 by rule 2 (Day 6 being 5 days after Day 1).
So (Day 1 + Day 2) + Day 3 = Day 1 + (Day 2 + Day 3) = Day 11.

9. Up to six potatoes and six carrots in the salad! What then?

Some people reading this might wonder what would happen if we used a lot more of the same ingredients. For example, in how many ways could you make the vegetable salad if you were allowed to put up to 6 potatoes and up to 6 carrots in it? Clearly there are 7 times 7 ways of doing it, as you could have

0 or 1 or 2 or 3 or 4 or 5 or 6 potatoes (7 possibilities) and

0 or 1 or 2 or 3 or 4 or 5 or 6 carrots (7 possibilities)

and as to each number of potatoes we could have seven different numbers of carrots (including zero carrots), so there must be 49 ways of making the salad, including the pineapple salad in which there are no potatoes nor any carrots! So to serve up all the possible salads (except the pineapple salad) you would need 48 days. Here is one way of serving all the possible salads in a cycle of 48, the first number referring to the number of potatoes, the second one to the number of carrots

01	42	06	35
44	13	33	64
24	52	53	25
23	10	54	60
56	04	21	03
32	22	45	55
16	12	61	65
30	15	40	62
05	63	02	14
66	51	11	26
36	43	41	34
31	50	46	20

where we start at the top left hand corner, go down the first column, then go to the top of the second column and go down this one, then go down the third column and finally go down the fourth column

A set of three days during which the total number of potatoes in all the three salads is either zero or a multiple of 7, and the total number of carrots in all the three salads is also either zero or a multiple of seven, can be called a triad of days. Now let us see whether we can detect some rules which will tell us, given any two days, which the third day is which would make the three days so obtained into a triad of days.. Let us say that it is allowed to have two helpings of salad, but then that day will have to be counted twice!

One easy way of finding such triads is to look at three days that are evenly spaced along the whole cycle. This would mean that there would have to be a “gap” of 15 days between any two consecutive elements of the days in the triad. Or put in another way, we shall get a day of our triad if we count to the 16^{th} day from any day of the triad.

If you have a second helping of salad, then you have used that day twice, and you need only look for one more day to complete the triad. You will find this second day if you leave a gap of seven days after the day of your double feast. For example, if you take the day on which you put 5 potatoes and 2 carrots into the salad, then if you "ignore" the following seven days, the eighth one will be one when you put 4 potatoes and 3 carrots into the salad. You will have eaten ten potatoes on the first day (two helpings!), and four potatoes on the next day of the "triad", which makes fourteen, indeed a multiple of seven. You will also have eaten four carrots on the day you had two helpings and three more on the second day of the "triad", which is seven carrots. So the days (52, 52, 43) , expressed in potatoes and carrots, form a triad.

There are several more ways of making triads of days. In what follows, the first number is the number of days between the first day and the second day of the triad, the second number being the number of days between the second day and the third day of the triad. Here they are, including the ones we have already seen:

(2 helpings, 7), (15, 15), (1, 11), (21, 2), (0, 3), (8, 5), (12, 6), (16, 9), (18, 10), and

finally : (23, pineapple), meaning that if there is a gap of 23, the third "day" is a pineapple party!

where the numbers refer to the day-numbers and not to the number of potatoes or carrots in the salad!

10. How do we "add" in such long cycles?

We can define "adding" of any two days as follows:

The sum of any two days is the day that makes up the triad to which the two days to be added belong. For example 19 + 20 = 24, as the "gaps" fall into the (0, 3) case.

We can check whether this "addition" is associative or not. Let us take the three days

19, 20, 21

to make things simple! We already know that 19 + 20 = 24 and 24 + 21 = 47, since we are now in the (21, 2) situation. But 20 + 21 = 25 as we are again in the (0, 3) situation, But we have: 19 + 25 = 10, as we are now in the (8, 5) situation. The numbers 10 and 47 clearly are not equal, so the "addition" is NOT associative.

But if we change the rules and say that the sum is not the missing day that makes up the triad, but 24 days after that missing day, I do believe we can "save" the associative rule!! Let us try and check it on the numbers 19, 20 and 21!

With the changed rules we have 19 + 20 = 24 + 24 days later = Day 48
because the days (19, 20, 24) are in the (0, 3) situation. :

but 48 + 21 = 38 + 24 days later = Day 14
because the days (48, 21, 38) are in the (16, 9) situation.
So (19 + 20) + 21 = Day 14

Now let us "bunch" the days the other way. That means finding 19 + (20 + 21)

We have 20 + 21 = 25 + 24 days later = Day 1
Because the days (20, 21, 25) are in the (0, 3) situation,

but then 19 + 1 = 38 + 24 days later = Day 14
because the days (19, 38, 1) are in the (18, 10) situation.

So 19 + (20 + 21) = Day 14

And we seem to get the same day, namely Day 14, as the "sum" 19 + 20 + 21, whichever way we do the "bunching". It seems that our addition "might" be associative!

There are some interesting observations one can make about the series of 48 salads.

For example take any three consecutive days' salads and take one plateful of the first day's salad, two platefuls of the second day's salad and three platefuls of the third day's salad. You will find that the total number of each vegetable is either zero or seven or fourteen, namely always a multiple of seven.

If you want to know today how many potatoes you are going to have in your salad tomorrow, all you have to do is to count the number of carrots in today's salad, multiply this number by 4, then subtract as many sevens as you are able to, and you will have tomorrow's potato ration.

If you want to know how many carrots tomorrow's salad will have in it, you just add the number of potatoes to the number of carrots in today's salad, multiply this number by 4, then subtract as many sevens as you are able to, and you will have tomorrow's carrot ration!

You might amuse yourself by trying to find the rule for what the salad will contain the day after tomorrow, or what it contained yesterday, assuming you only know the composition of today's salad. You might have noticed, for example, that you get yesterday's number of carrots by multiplying today's number of potatoes by 2 and then subtracting any possible sevens. But what is the "rule" for yesterday's number of potatoes? Try to find it!

11. PROBLEMS

Make up a cycle of 24 days, with the rule that you must always use less than 5 potatoes as well as less than 5 carrots in any salad. Then find the triads in your cycle. The total number of potatoes and the total number of carrots altogether on three days of a triad must be a multiple of 5.

What about addition? Will it be associative? Or do you have to "save" it? If so, by how many days ahead do you have to go to make your addition associative?

BRAINTWISTER. If you want a hint about how to make a cycle of 80 days, a different salad being consumed on each of the eighty days, then read on. You will need to have potatoes, carrots, onions and turnips handy. Of each vegetable you are allowed to put either none or one or two pieces into the salad. In order to make an 80 day cycle, follow these rules:

Number of carrots in today's salad = number of potatoes in tomorrow's salad

Number of onions in today's salad = number of carrots in tomorrow's salad

Add the number of potatoes to the number of turnips in today's salad, but subtract 3 if you get more than 2 as the sum = number of onions in tomorrow's salad

Number of potatoes in today's salad = number of turnips in tomorrow's salad

You should find that 40 days after any particular day, the salad for that day will be obtained from today's salad by replacing 1 by 2, 2 by 1 and zero of any vegetable will remain zero of that vegetable.

If you are curious enough to look into finding the sets of three days during which you will have had either none or three or six pieces of each vegetable, putting the salads all together from the three days, you should find the following "pairs of gaps", the first number being the number of days between the first and the second day, the second number being the number of days between the second and the third day:

(35, 0), (15, 1), (27, 2), (3, 8), (13, 4), (5, 25), (16, 6), (20, 7), … … …

You could check if the above are correct, and find all the other possible ways of putting three days together such that on those days you will have consumed, altogether, a multiple of three of each kind of vegetable!

If you have difficulties, contact me at zoltan@zoltandienes.com