Bruce and Alice learn about ratios
(don’t forget that numbers are in base eight!)

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One day the teacher treated the children to another surprise announcement. He said, as though he was announcing something really important:

"This week we are going to learn about ratios. You will need a lot of green and red counters, and some paper and black pencil and a red pencil for writing down what each ratio is going to do. Arrange yourselves in groups of four, and make sure you have a lot of counters and quite a large number of small sheets of paper. Pages out of a pad will do very well"

The children procured all the material and sat down in their groups and awaited the grand announcement about RATIOS!

"Does anyone here know what a ratio is?", asked the teacher.

Not one hand went up. Clearly the teacher was going to have a tough job!

"How many boys and how many girls are there in this class?", asked the teacher, after a short period of silence.

"1 2 boys and 1 7 girls!", said the children almost at once

"Don't forget, Bruce and Alice, that when we say 1 2, it is you would call ten at home, and when we say 1 7, this you would call fifteen, is that right?"

"That is correct", replied Bruce, "But we have got used to thinking in eights by now, so please don't trouble to translate for us!"

"Bruce, Alo, Alice, Ata and Unta, please come to the front", said the teacher.

Our five friends duly did so and wondered what they would have to do next. The teacher said: "We have here in the front 2 boys and 3 girls. Please call two more boys and three more girls to stand together in another group".

Bruce called two boys and three girls and stood them in a group by themselves.

"How many more times can you do this?", asked the teacher, "and would there be anyone left over?"

"No!", said Alice "because we could make five groups, each group having two boys and three girls in it!"

"That is correct", said the teacher, "So for every two boys, we can find three girls in this classroom. We say in such a case that the RATIO of the boys to the girls is as 2 is to 3: Now go back to your seats and write this sentence :WE MUST TRY HARDER. Can anyone tell me the RATIO of the number of vowels to the number of consonants in that sentence?"

"I think I know", ventured Ata, "I think it is as 1 is to 2!"
"That is quite correct", said the teacher approvingly, "Can you tell us how you came to that conclusion?"
"Well", replied Ata. "It's like this: e → w m, u → s t, y → t r, a → h r, e → d r,
so it seems that each vowel has two consonants to go with it, so it is one vowel to two consonants. Is that correct?"

"That is absolutely correct", replied the teacher, "Now can anyone tell me the RATIO of the number of girls to the number of ALL THE CHILDREN ?"

"Easy", said Alice, "We have seen that there are three girls in each group of five children, so the RATIO is as 3 is to 5! I am sure that is correct! And of course there are two boys to every five children in the class!", added Alice triumphantly.

"Now you can start working with your counters," said the teacher, "The sheets of paper are there for you to write ratios on. If you use a black pencil, then you have to put the same color counters on each side of your sheet. If you use a red pencil, one side must have counters of a different color from the other side. Just put a pile of counters on one side of your sheet, then another pile on the other side. Then write the ratio on your sheet that the number on the left is to the number on the right."

"I don't understand how I can write the ratio, just by looking at the piles", said Unta.

"I think I know", said Alice, "Why don't you put 3 0 green ones on the left and 2 0 green counters on the right. You could then make a green pile of two on the right, for every green pile of three on the left. So the ratio would be 2 on the right for 3 on the left, and you would use a black pencil, since both piles are green. You would also use a black pencil if both piles were red!'
Unta, and some of the other children, quickly made the piles as suggested by Alice and they soon realized that indeed there was a "2 on right for every 3 on left" ratio that connected the two piles. They wrote "2 for every 3" on the sheet between the piles.

The teacher walked around the class and helped one or two children to make the piles, and when he thought that all the children had understood Alice's suggestion, he said:

"Now make your own piles, and write the ratios on the pads!"

These were some of the pairs of piles that the children made:

2 0	→	1 4	3 0	→	2 4
o o o o	3	o o o	o o o o o o	5	o o o o o
o o o o	for	o o o	o o o o o o	for	o o o o o
o o o o	every	o o o	o o o o o o	every	o o o o o
o o o o	4	o o o	o o o o o o	6	o o o o o
red	green	red	red	red	green

3 0	→	2 2	6 2	→	2 4
o o o o o o	3	o o o o o o	o o o o o o o o o o	2	o o o o o o o o o o
o o o o o o	for	o o o o o o	o o o o o o o o o o	for	o o o o o o o o o o
o o o o o o	every	o o o o o o	o o o o o o o o o o	every	
o o o o o o	4		o o o o o o o o o o	5	
			o o o o o o o o o o		
green	red	red	green	green	green

Bruce and Alice did not forget that 2 0 meant sixteen, that 1 4 meant twelve, that 3 0 meant twenty four, that 2 4 meant twenty, that 2 2 meant eighteen, that 6 2 meant fifty, since all the numbers were written in eightlandish manner!

When the children had all finished making their "ratio piles", the teacher asked a few other questions, such as:

"A week, to the number of days in it is as what to what?"

"The number of shoes on your feet to the number of children that you are, is as what to what?"

"If in an exam, out of 24 questions, you were able to do 18, what is the ratio of those you could do to those you could not do? What is the ratio of those you could not do to the number of all the questions?"

At this point the bell went, so the teacher asked the children to bring back five examples each of ratios from their own experience, as homework.

"Can you think of five ratios?", asked Alice, turning to Bruce.

"Sure I can", replied Bruce "Make a fruit salad with two ornages for every three apples.Throw a die, you have one chance out of three of getting more than four. Enough ?"

"I got four out of five correct in my last test!", added Alice, "There are three vowels out of thirteen letters in the alphabet! There is one even number out of every two numbers! We eat three meals on any one day!" she added "It's easy. We have never had such an easy homework!"

"I think I know the secret of finding out the ratio in which a pile is to another pile!", said Bruce to Alice as they were going into lunch with the other children. "You have to divide up each big pile into smaller equal piles. In each of our two big piles, whose ratio to each other we want to find, we have to make exactly the same number of smaller piles. Then you look at the sizes of these smaller piles and that tells you the ratio!"

"Show me with these sugarlumps", said Alice to Bruce as they sat down to a table.

"Look", said Bruce "I can't waste so much sugar, but I will draw it on these paper napkins."

He drew forty eight crosses on one napkin and sixty crosses on another and he said: "There are 6 0 crosses here (or forty eight as we would say at home!) and 7 4 crosses on the other napkin (or sixty, as we would say at home). Let me draw circles round every set of four crosses on the first napkin, and also circles round every set of five crosses on the other napkin. There are twelve circles on each napkin. So the number of crosses on the first napkin is to the number of crosses on the second napkin, as 4 is to 5. Hi dot kapta? (which means "Have you understood me?" in Ruritanian!)"

"I see", said Unta, who was sitting next to them and listening to Bruce's explanations. "You have made the same number of equal smaller piles on each of the napkins, and comparing the smaller piles it is easy to see that for every four crosses on one napkin, there are five crosses on the other napkin! That is really quite easy! After lunch I shall have to make up my five ratios for my homework!" concluded Unta.

By the time they were going into class the next day to have their mathematics lesson, every child was quite at home about what was meant by the ratio of one amount to another amount. The teacher looked at all the homework that had been brought in and was pleased to find that there was nobody who had not grasped the idea of a ratio. Then he said this to the class:

"Today we are going to try and put two ratios together and out of them, make another ratio. Let me call the pile you place on the left of your sheet the INPUT, and the pile on the right of your sheet the OUPUT. The input goes "into" the ratio; the ratio, rather like a machine, does something to it, and then out comes the output on the right. We shall try to use this output as an input for another ratio, and see what comes out on the right of this second ratio."

Bruce and Alo, who were sitting together, were already busy drawing their two ratios on their sheets of paper. These are the two which they agreed upon, both drawn with black pencils :

	2 for every	output		3 for every	output
Input	3		input	4	

"Let us use all green counters", suggested Alo, "and maybe we could try 1 1 counters as the first input. You then get 6 green counters as the output"

"Now we have to use this as the input for the second ratio", said Bruce, "but we can't, since although we can put 3 out for the first 4 in the input, what do we do with the two counters that are left over which we ought to use as part of the input of the second ratio?"

The teacher was overhearing this conversation so he suggested:

"Why not try another input for the first ratio?"

"I wonder how many we should have?", said Alo, "we must come out with sets of four as the ouput of the first ratio. How about 3 0 (twenty four in the English system, whispered Alo to Bruce) ?"

"We can make 1 0 sets of threes from our input, so we should have 1 0 sets of twos at the output of our first ratio, which is of course 2 0", said Bruce. "And there are four sets of fours in 2 0. If we place a set of three for each one of these sets of four, we shall get 1 4 as the output on the right of our second ratio!", concluded Bruce.

"Do you know what?", chimed in Ata, who was watching the proceedings, "You seem to have made a ratio that gives you one at the output of the second ratio for every two at the input of the first ratio!. There are just 1 4 sets of two in a set of 3 0, and there are 1 4 sets of one in a set of 1 4!" concluded Ata.

"You are right!", said Bruce, "but would we have come up with the same ratio if we had started with another input?", added Bruce questioningly.

"You are right", admitted Ata, "We ought to check several inputs and see what happens!"

"Let's try 4 4 as the first input", suggested Ata ."That is what you call thirty six", she said, turning to Bruce. "There are 1 4 sets of three in a set of 4 4 counters, so our first output must be 1 4 sets of two. This comes to 3 0. Now we have to use these 3 0 counters as an input to our second ratio, which is a 3 for every 4 ratio. Now, how many sets of four counters are there in a set of 3 0 counters?"

"There are six sets of four counters in a set of 3 0 counters, so there must be six sets of three counters in the output of our second ratio", added Bruce. "So the final output must be 2 2 counters!"

"So from the initial input of 4 4, we get a final output of 2 2", concluded Ata, "so from 2 2 sets of two counters, we get 2 2 sets of one counter each! Again 1 for every 2, as before!"

"It seems that it's working", chimed in Alo.

The children did not even notice that the teacher was standing behind them, watching them and listening to what they were saying! He tapped Alo on the head and said to him:

"Would you like to explain what you have found to the class?"

Alo agreed and carefully went through all the steps that they had been through in their little group, making sure that nobody was left behind. If there was anyone left with a perplexed expression, he just went over the steps again, very slowly, so that nothing could be missed.

"I would like all the groups to choose two ratios and put them together as we have put the 2 for 3 and the 3 for 4 ratios together here. Then I would like each group to find the single ratio that comes out of putting the two ratios together", suggested the teacxher.

"Can we put more than two together", asked Alice.

"Sure thing", replied the teacher, "You can try and put three or even four ratios together, each output serving as the input for the next ratio. You could also think of things that you could have been doing that would have resulted in such chains of ratios!"

The children got down to the serious work of constructing their chains. Bruce, Alice, Ata, Unta and Alo got together as a group and thought up the following ":story" that might have happened:

In a class there were 2 boys out of every 5 children. Only 3 boys out of 4 went to skating lessons, and 5 out of 6 of the skaters were chosen to do a skating performance. How many children, out of all the children in the class, performed?

After a large number of first outputs that they tried, they came upon the number 5 0 (namely forty children) that could have been the number of children in the class. If 2 out of 5 were boys, then there were 2 0 boys (sixteen in English numbers) in the class. There are four groups of four boys in a group of 2 0 boys, so out of each group of four, only three learned to skate. So there were 1 4 skaters, which meant two sets of six. If 5 out of 6 skaters were chosen to perform, then 1 2 boys (ten in English numbers!) did the performance. So there were 1 2 performers out of 5 0 children, which is 1 performer for every 4 children in the class. The ratio constructed out of the chain of three ratios is the ratio 1 for every 4.

"I want to make a story for the girls as well", said Alice.

"Go ahead", said the teacher, "but make it a little different!"

"All right", replied Alice. "There are 3 girls out of every 5 children, so the first ratio is 3 for every 5. Let us suppose that 2 girls out of every 3 girls learn to skate and that 3 out of every four skaters are chosen to perform for the school. There must have been 3 0 girls in the class (twenty four, as we would say at home), and 2 out of 3 of these would give us 2 0 skaters, which would make 4 groups of 4 skaters (2 0 is sixteen in our language!). So there will be 4 groups of 3 skaters performing from the girls' part of the class. This is 1 4 performers (twelve, in our language). If I am not mistaken, this means that 3 girls out of 1 2 children in the class performed"

"Altogether how many children out of how many performed, counting the boys and the girls together?", asked the teacher. "Clearly 1 2 + 1 4 = 2 6, so 2 6 children out of 4 0 children performed"

"Let me think", replied Alice, "So there must have been 1 3 out of every 2 0 children in the class who performed at the skating performance. We would have said eleven children out of every sixteen children performed, the ratio being eleven to sixteen. That is if they have ratios where we live as you have them in Ruritania!"

"Can I show you another one"" asked one of the children.

"Be quick, it is nearly time to go!", cautioned the teacher.

"Here we go!", said the child, "In my class last year there were 2 girls out of every 3 children. I remember that 3 out of every 5 girls learned music, and 1 out of 2 of those who learned music, played the flue. I think 1 child out of every 5 in the class played the flute!"

"How do you work that out?", asked the teacher.

"You said I must be quick, so I told you the answer right away!", replied the surprised child. "There were 3 6 children in the class, 2 4 out of these were girls. Then 1 4 girls learned music, 1 out of 2 of whom learned the flute, which makes 6 flute players. There are 6 sets of 5 children in a class of 3 6, and there are 6 sets of a single child, in a set of six flute players. So that makes 1 flute player for every 5 children in the class", concluded the child.

"That was quick thinking", said the teacher, just as the bell went. "Make up some more stories for homework, make them as long as possible!", he concluded.

At the next lesson on ratios the teacher had something new for the children. He said to them:

"There is another way of combining ratios, which I would like you to learn. But we shall have to pay more attention to the color of the counters that we use to represent ratios. In fact in a pile of green and red counters we shall only look at the difference between the number of green ones and the number of red ones. If there are more green than red counters, we shall call such a pile a more pile. If there are less green counters than red counters, we shall call the pile a less pile. If the pile has just the same number of green counters as red counters, we shall call it a zero pile, since in this case the green are not more than the red nor less than the red"

"I have made a green pile, a red pile and a zero pile", Bruce said to the teacher, "Look, here they are!"

These were the piles Bruce had made:

g g g g g r r   g r r r   g g g g g r r r r r

"Yes, thank you Bruce" said the teacher, looking at the piles, "the first one is a 3 more pile, the second one is a 2 less pile, and the third one is indeed a zero pile!"

"Are you suggesting", asked Alo, "that when looking at a pile, we should ignore how many counters there are in it altogether, but only think of how many more or less green counters there are in it than red counters?"

"That is exactly what I mean," replied the teacher. "Look at it this way", continued the teacher, "suppose you have a bank account and you pay in some cheques and you write some other cheques for paying for things. The amounts you pay in can be shown by green counters, those that you draw out by writing cheques, can be shown by the red counters. When you look at your balance, what you really want to know is what you have in the bank, or what you owe to the bank, in case you have overdrawn! A balance in your favour will be shown in black, any amount owed will be shown in red. Has your father ever told you he was in the red at the bank? Next time, or if he ever tells you that, you will know that he has written cheques for more than the cheques that he has paid in!"

The children appeared to be satisfied with this explanation. So the teacher now went on to explain the new method of putting ratios together:

"Make two ratios and then use the same input for both ratios. You will get two outputs. Then add these two outputs. This sum will be the final output. Now try to find a single ratio that will give you this final output, when you give it the input you used for your two ratios"

He put this example on the board, to make sure the children had understood the new way.

In a school you can take French or you can take Spanish as a second language, but you cannot take both. In one class 1 child out of 5 decided to do Spanish, and 2 children out of 3 decided to do French. Some decided not to do a second language. How many children out of how many decided to take a second language? How many out of how many did not want to learn a second language?

"Maybe 3 6 children in the class would be a good number. Don't forget, Bruce and Alice, that would be called thirty where you live!"

The children all wanted to solve the teacher's problem first, before setting themselves any others. Alice was very quick and said quickly to Bruce:

"That's easy! Obviously only 6 children did Spanish and 2 4 children (meaning twenty!) did French, so if you add those you get 3 2 children (twenty six in our usual numbers). So 1 5 children out every 17 children did a second language (in other words thirteen out of fifteen), and 2 out of 1 7 children did not (in other words two out of fifteen)!"

"Now let us make up our own ratios to combine", said Ata, who was sitting at the same table as Bruce and Alice.

"But we have not drawn the ratios for the teacher's problem on our sheets", objected Alice.

"All righ, then", agreed Ata, "here they are! It's very simple!"

And she drew these ratios on her sheets of paper:

	1	OUT(1)		2	OUT(2)		1 5	OUT(3)
	for			for			for	
	every			every			every	
IN	5		IN	3		IN	1 7	

OUT(1) + OUT(2) = OUT(3)

"Are you happy now?", asked Ata, turning to Bruce and Alice.

"All right, I suppose it will do", Bruce replied, "Now let us invent our own. We could just do it with the counters. Then we can use green ones as well as red ones and see what happens!"

"I would like to try these", said Alice, drawing these two ratios on some paper:

2 for every 3	1 for every 6
green	red

"I am going to try 3 0 green counters as the input, which will count as a 3 0 more pile, as there are 3 0 green ones and no red ones", said Alice. "That will give us 2 0 green counters as our first output. There are four piles of 6 counters in a pile of 3 0 counters, so each pile of 6 green ones will give us one red one at the output. So the second output is 4 red counters, right?"

"Now we have to add these. In the joint pile there will be 1 4 more green counters than red counters, so it will count as a 1 4 more pile", volunteered Unta, "There are 1 4 piles of two in a 3 0 pile and 1 4 piles of 1 in a 1 4 pile, so the third ratio we are looking for is the green ratio 1 for every 2.

"I have been looking ahead in the next chapter of our mathematics book", said Alo ,"it says that instead of writing "for every" all the time, one could just draw a line. And what we have been doing today is called adding ratios, and we could just put a plus sign between the ratios and the equal sign to link them to the third ratio we have been finding"

"That sounds quite simple", agreed Alice, "But what is the first kind of way called that we used for putting ratios together?"

"That is called multiplying, and we can either use just a dot or an X, as we do in normal number work when we multiply!", replied Alo.

"But what about the green and red writing?", asked Ata.

"I am not sure about that", replied Alo, "I believe when it is a red ratio, then instead of a plus sign, you write a minus sign, but we had better ask the teacher, it's no good rushing into it!"

They agreed that they now knew how to write down the multiplications and the additions of green ratios, but they definitely were waiting till the next lesson before they would start writing their red ratios in the "new" way! When Alo was left by himself, he could not help himself, he just had to try and write these things down. This is what he wrote :

+ (1/5) + (2/3) = + (15/17)   + (2/3) - (1/6) = + (1/2)

(+ 2/5) x (+ 3/4) x (+ 5/6) = (+ 1/4)   (+ 2/3) x (+ 3/4) = (+ 1/2)

but with the multiplications he was not sure how to put the pluses!
You can now invent the next lesson!