Bruce and Alice learn about factors.

One day the mathematics teacher greeted the children with this announcement:

"I have had special permission from the Ministry of Numbers of the State of Eightland, to use numbers as they do in Tenland. That should be very easy for Bruce and Alice, but you others must be careful now to think of the number 1 2, which we have been calling one two, as ten, which will now be written as 1 0. If you have any problems, I am sure Bruce or Alice will help you out!"

"We sure will", said Bruce and Alice together, "So what is the lesson for today?", they asked.

First let us do some multiplying by 2. This is also known as DOUBLING. Alice!, you start with the number 3, and go on doubling. Write your numbers on the board!"

Alice came to the board and wrote this:

3, 6, 12, 24, 48, 96, 192, 384, and then stopped.

"They are getting too big to do in my head after this!", admitted Alice.

"Bruce, you do one, but start with the number 9. You know, the number we usually call 1 1!", said the teacher. Bruce went to the board and wrote these numbers,

9, 18, 36, 72, 144, 288, 576, and he stopped here as he thought it was enough.

"Now, Unta, come and start with 1, and then write Alice's numbers and then Bruce's numbers underneath!", suggested the teacher.

This was the result of what Unta had written on the board:

1 2 4 8 16 32 64 128 256 512 . . .

3 6 12 24 48 96 192 384

9 18 36 72 144 288 576

27 54 108 216 432 864

"Why did you write the fourth line", asked the teacher.

"Well, I thought", said Unta somewhat hesitatingly, "in the second line the numbers were 3 times the numbers in the first line. In the third line the numbers were also 3 times the numbers in the second line. So I thought I would make the numbers in the fourth line 3 times the numbers in the third line"

"That was very good", said the teacher, praising Unta's correctly applied intuition. "Now all have a good look at the numbers on the board, and see if anything else comes to you!", he said, speaking to the class in general.

"I can see something", said Bruce, "Can I say?"

"Carry on!", said the teacher.

"If you take any square shape of four numbers, the one on the lower right is always 6 times the one on the upper left!"

All the children checked this, all over the sequences of numbers, and they saw that what Bruce had said was in fact correct!. Then Ata chimed in:

"Can I say something else?"

"Tell us!", said the teacher.

"If you look at the next number but one of any number, it is always 4 times this number!"

"I can see a 9 times!", interjected Alo, "If you move down, jumping over a number, you get 9 times the number from which you have jumped!"

"I wonder if anyone can see a times 5 in the table?", asked the teacher.

They looked for a long time, but nobody could see anything to do with fives. Then the teacher suggested that they could also be looking for sums of numbers! At this Alice put her hand up and when called by the teacher said:

"If you want 5 times a number, look at the little square of which your number is the upper left one. Then if you add the upper right number to the lower left number of this little square, you will get 5 times your chosen number!"

"And if you want a 7 times trick", chimed in Bruce, "All you have to do is to take two square shapes of numbers, one to the right of the other. This will be a rectangle. Then 7 times the number at the upper left of this rectangle, will be the sum of the upper right number and of the lower left number of the rectangle!"

Ata though that they should put in these numbers into the scheme as well. They finally decided that they would put in just the 5 times numbers, as otherwise the whole thing would have become too difficult to look at!. This is what Ata wrote on the board:

1	2	4	8	16	32	64
5	10	20	40	80	160	
3	6	12	24	48	96	
15	30	60	120	240		
9	18	36	72	144		
45	90	180	360			
27	54	108	216			
135	270	540				
81	162	324				
405	810					
243	486					

All the children were so pleased with this way of organizing the numbers that they all made careful note of them in their books, although some of them wrote the numbers in eightlandish manner!

This meant that some of the children had written the first row as

1 2 4 10 20 40 100 200 400 1000 …..

which they thought was much easier to remember than the tenlandish way of writing them!

"Can somebody see how you find 8 times a number in these tables?", asked the teacher.

"Oh yes", said one of the local children who had written the numbers down in the eightlandish manner, "You move to the right and jump over two numbers. You always get a zero tacked on at the end, as we do when we multiply by eight, or by 1 0, as we usually say!"

"So it is only the 5 times and the 7 times that need any adding, all the rest of the multiplications are there already, we just have to know which way to go to look for the product", said one of the other local children.

"So if anyone can do a 2 times and a 3 times, he can immediately also do a 4 times, a 6 times, and 8 times and a 9 times!", exclaimed Bruce, "What a smart way to remember the multiplication facts!", he added

"I should like you to compare these words, most of which are English words, but not all of them are, with Ata's table of numbers. Here they are" said the teacher, writing the "words" on the board:

ball	fall	gall	hall	tall
bat	fat	gat	hat	tat
bell	fell	gell	hell	tell
bet	fet	get	het	tet
bill	fill	gill	hill	till
bit	fit	git	hit	tit
bull	full	gull	hull	tull
but	fut	gut	hut	tut

Alice was quick to notice the alphabetical order in the "words" and said:

"When you move up the alphabet for the first letter of the word, you are multiplying by 2. When you move up the vowels, you are multiplying by 3. When you move from ll's at the end of a word to a word ending in t, you are multiplying by 5!"

"So to multiply by 9, you change a to i   or else e to u", added Bruce, "And to multiply by 4: b becomes g, f becomes h and g becomes t"

"To multiply by 6 all you have to do is to move up the vowel as well as the first letter!", added Unta.

"So if you want to know the number of a word, you have to remember these values", suggested Ata," you have to know that b = 1, f = 2, g = 4, h = 8, t = 16 (but only at the start of a word), t at the end of a word is equal to 5!. Then you also have to remember that a = 1, e = 3, also i = 9 and u = 27. Then you multiply these numbers and that is the number of your word!"

"For example gut = 540, because g = 27, u = 4 and the final t = 5 and multiplying these three numbers we get 540!", suggested Alice. "So we can have a secret code in telling each other how much money we have! We use one of these words!"

"Unless you are unlucky enough that you need a 7 for working out your money!", said Bruce. "What is the word for the number 14?"

"We would need to think of another way to end a word, which is neither a double l nor a t", replied Alice. "Maybe they could end in n for a 7-word!"

At the next mathematics lesson the teacher wanted them to make "maps" of certain words, together with the numbers that went with them. For example the "map" of the "word" fet, which has the number 30 going with it, would look like this:

	2	6	
	fall	fell	
1	3	10	30
ball	bell	fat	fet
	5	15	
	bat	bet	

where the change from b to f is "going uphill" and is a multiplication by 2; where the change from a to e is a step to the right and is a multiplication by 3 and where double ll to t is "going downhill" and is a multiplication by 5. It was soon clear to the children that all these numbers were somehow made up of the three numbers 2, 3 and 5. The "column" just before the 30 was always the product of two of these three numbers. And 30 was the product of one number in the first column by another number in the second column.

"On this model we can make the map of 42 as well as the map of 70, or even of 105", suggested Bruce.

"How would you do that", asked Alice.

` "Just replace the 5 by a 7. Then going downhill becomes a multiplication by 7, the other steps remain the same. So instead of 5, 10 and 15, we would have 7, 14 and 21, and of course instead of 30, we would have 42!", replied Bruce logically.

"Let me see if I can do the 70. " said Alice, "The downhill can stay for the times 7 as in the map for 42. There will be no times 3, so the step to the right could now be a times 5. The going uphill can stay as times 2. Is that right, Bruce?"

"You had better ask the teacher, I am only just working these things out myself", said Bruce. "Maybe you could try the 105 before you ask him!"

Ata and Alo were already working on mapping the number 60, while Unta was busy with the number 72. This is what Ata and Alo came up with:

		3		6		12		
		bell		fell		gell		
1	2		4		15		30	60
ball	fall		gall		bet		fet	get
		5		10		20		
		bat		fat		gat		

On this map a step to the right is a multiplication by 2 and the moving on of the first letter to the next letter used in the alphabet. Going uphill is multiplying by 3, and a change of vowels for the words. Going downhill is a multiplication by 5 and a change in the ending from double l to t for the words.

Here is Unta's map of the number 72, which has the word "hill" to go with it

1 ball	2 fall	4 gall	8 hall
3 bell	6 fell	12 gell	24 hell
9 bill	18 fill	36 gill	72 hill

Here the multiplications and the changes in the words are all quite clear

The children decided to play the following game:

One child picks a word. The other player has to reply with a chain of words, beginning with the word BALL, reaching the word picked. The child who is solving the problem must say what is being changed from each word to the next. Only three types of change were allowed:

(i) change the first letter to the next one in alphabetical order,
(ii) change the vowel to the next vowel in alphabetical order,
(iii) change the ending from double l to t.

For example if the word TIT was picked, one possible answer would have been:

BALL (iii) BAT (ii) BET (ii) BIT (i) FIT (i) GIT (i) HIT (i) TIT

Which makes seven changes that need to be carried out one after the other. The corresponding problem for numbers would have been to pick 720 and start from 1.
A solution might have been

1 (x2) 2 (x2) 4 (x2) 8 (x2) 16 (x5) 80 (x3) 240 (x3) 720

Note that although TIT has the number 720, the two solutions do not follow the same path. However, they both use seven steps.

It soon became a favorite pastime with the children to play such games. Multiplying by 2 and by 3 soon became like second nature to them, and they could reel off sequences of numbers, doubling each time to get the next number, or multiplying by 3 each time to get the next number.

At the start of one mathematics lesson the teacher announced:

"Today we are going to learn about squares. I want each group to have a fair number of little squares, as well as of little triangles. Make sure all the squares you take are the same size. Also make sure that all your triangles are the same size. Then I shall want you to make larger squares with your little squares, and larger triangles with your little triangles. Each time note how many pieces you have used in your construction"

The children soon got busy and made a lot of squares and a lot of triangles. Some made some very big ones, using as many as 169 pieces! These were the numbers that they recorded, both for the squares and for the triangles:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, …….

"I have noticed something!", said Alice to the teacher, "In the triangles, all the rows have an odd number of triangles in them!. The top row just has one, the next row has 3, then the next row has 5 and so on. So if we add up all the odd numbers up to a certain one, we shall always get one of the numbers we have found!"

"Oh yes!", chimed in Alo, "you can do the same with the squares! If you start at the top left corner, there is one square. Then you move to the right and go down diagonally towards the left, you find 3 squares. Next time you find 5 going down from the next square to the right, but going down diagonally as before! I wonder if there is another shape that would do the same thing!", continued Alo, thinking aloud.

"Why don't we try those trapezoids?", suggested Bruce.

"What are trapezoids?", asked Alice.

"They are quadrilaterals with two sides parallel", replied Bruce.

"I am none the wiser", retorted Alice in a somewhat cross tone.

"See those pieces that look like roofs you might see on houses?", said Bruce patiently, pointing at some material on the shelves. "The topmost part of the roof is parallel to the lower part. What is more, the two sloping sides and the upper side are all equal in length and the lower side is twice as long as one of those three", Bruce went on explaining, "How many roofs would you need to make another roof?", he asked.

"I will try and see", said Alice and went to collect some "roofs". She soon saw that if she put one roof down, and then put another one on top of it but upside down, she could fit two more in, one on each side, and so make another, bigger roof. To make it bigger still, she needed five more roofs. To make it even bigger, she needed seven more. She soon realized that she needed the same number of roofs to make bigger roofs as she had needed triangles to make bigger triangles or squares to make bigger squares! She was quite engrossed in making quite enormous roofs when she suddenly heard the teacher say:

"I would now like you to get some cups and some small pebbles from the shelf", said the teacher, "Choose a number; put this number of cups in a row in front of you. Then into each cup, put your chosen number of small pebbles. Then count how many pebbles you have used altogether. Note this number down in your book. Then choose another number and do it again. When you have used five or six different numbers, compare these numbers with what you found with the squares and the triangles, or even with the roofs, as Alice calls them"

The children procured their material and worked on filling their cups as suggested by the teacher. It was not long before they all realized that they were getting exactly the same numbers as the ones that they came upon when working with the squares and the triangles

"The square numbers are not only square numbers", ventured to put in Bruce, "they are also triangle numbers, also pebbles in cups numbers!"

"And roof numbers!", added Alice.

"We have got used to calling them square numbers", said the teacher, "So that is what we shall call them. I should like you to learn all the square numbers right up to 400 as home work for the next mathematics lesson", added the teacher. "But now we could look at the square minus one numbers. Ata, look at one of your big squares you have made. Take one little square away from one of the corners. "Ata obligingly did so, "Now make your little squares into a rectangle, without any bites taken out of it, as your big square now has in it!"

"Does your rectangle have more rows in it than the square had or less?", asked the teacher.

"Less", replied Ata, "because I removed the top right little square, and moved the whole top row to the left and turned it into a column! So the rectangle has one less row in it than did the square, but it has one more column than did the square, to make up for it!"

"So if you removed one square from a 20 by 20 square, namely from a total of 400 squares, how many rows and how many columns would you have?", asked the teacher.

"Nineteen rows and twenty one columns!", said Bruce and Alice at once, who were quicker than the rest with tenlandish numbers. "That means that 19 times 21 is equal to 399, does it not?"

"You have hit on the reason why I have asked you to study not only square numbers but also square minus one numbers!", said the teacher to Bruce. "Can you tell me right away what 12 times 14 must be?"

"It must be 168", replied Bruce, "because 13 times 13 is 169, and one less than that is 168"

"You see", explained the teacher, "If you know your square numbers, you can multiply any two numbers whose difference is 2, as long as they are not too big!"

“Could we remove more than one little square from our big square?”, asked Alice.

“Yes, of course you can”, replied the teacher, “but you might not always be able to make a rectangle with the little squares you have left!”

“How many can we remove then”, insisted Alice, “so that we can again make a rectangle out of what is left?”

“Try removing four little squares”, suggested the teacher.

“From where shall we take them”, asked Ata.

“You can take them from where you like”, replied the teacher.

“I am going to take one from each corner”, said Alice

Alice did so. This left her with the top row two little squares short. The bottom row was also two little squares short. So Alice removed these two shortened rows, turned them round and made them into columns. Having removed the two shortened rows, the columns happened to be exactly as long as the two “new” columns! So Alice slid these columns along, putting one on the left and one on the right of the remaining squares. This made the rows two little squares longer than they had been in the square!

“Can you tell us what you have made, Alice?”, asked the teacher.

“Oh yes”, answered Alice happily, “I have now two less rows than I had at the start and each row has two more little squares in it than at the start. So if I had had 400 little squares, I would now have 396, and there would be 18 rows of 22 squares each, so 18 x 22 = 396!”

“Great!”, said Bruce, “So how much is 13 x 17?

“That’s simple”, replied Alice, “It must be 221, since 15 x 15 = 225, and if you take away 4, you will have 221, and you will have 2 less rows than 15, and 2 more than 15 in each row!”

“Now you can multiply any two numbers whose difference is four,” said the teacher “as long as you know the square of the number half way between the two numbers!”

“Of course”, said Bruce, “for example, since 25 x 25 = 625, it must be true that 23 x 27 is 621!”

“For homework”, said the teacher, “You can work out what happens when you take 9 away from a square!”

1		**2**		**4**		**8**		**16**		**32**		**64**
	5		**10**		**20**		**40**		**80**		**160**	
3		**6**		**12**		**24**		**48**		**96**		
	15		**30**		**60**		**120**		**240**			
9		**18**		**36**		**72**		**144**				
	45		**90**		**180**		**360**					
27		**54**		**108**		**216**						
	135		**270**		**540**							
81		**162**		**324**								

x 2 go to the next number on the right

x 3 go to the first number downwards

x 4 go to the second number on the right

x 5 if the multiplicand is not a multiple of 5, go diagonally right-down

x 6 go to the second number diagonally right-down

x 8 go to the third number on the right

x 9 go to the second number downwards

x 7 go to the second number on the right as well as to
the first number downwards and add them.