# STRIP MATHEMATICS

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The New Zealand Mathematics Magazine, Vol 39, no.1. (May 2002)

Mathematical fun without numbers, letters, formulae or equations, Part II.
The New Zealand Mathematics Magazine, Vol 39, no.3. (December 2002)

Mathematical fun without numbers, letters, formulae or equations, Part III.
The New Zealand Mathematics Magazine, Vol 40, no.1. (March 2003)

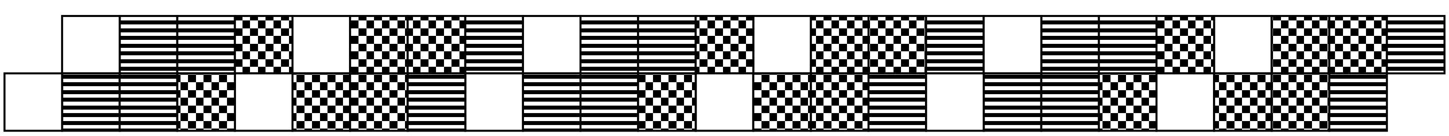
Mathematical fun without numbers, letters, formulae or equations, Part IV.
The New Zealand Mathematics Magazine, Vol 40, no.2. (July 2003) pp. 42-47

A great deal of mathematics involves cycles that go on repeating indefinitely. As often as not one has to consider such cycles superimposed on each other. A cycle can be superimposed on the same cycle simply by starting the cycles at different points. So we could have two, three, four or more of the same cycle running “at the same time” but with a phase difference.

I shall call the elements that repeat colours. They could equally well be shapes, or letters or numbers. I shall start with just two colours and finally work up to cycles of five colours.

Our sequences of colours will be drawn on strips. Two or more strips with the same sequence of colours can then be placed parallel to each other and the strips can be moved to the right or to the left thus creating “columns” of colours

Here is an example of two strips the top one having been slid along the other by one space. Here we are using three colours



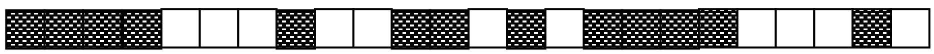
It will be seen that some interesting mathematics can be studied in this way, without the use of any complex system of symbols or equations

I shall start with two colours, using four strips, after which I shall restrict myself to just two strips, but I shall use three, then four and finally five colours.

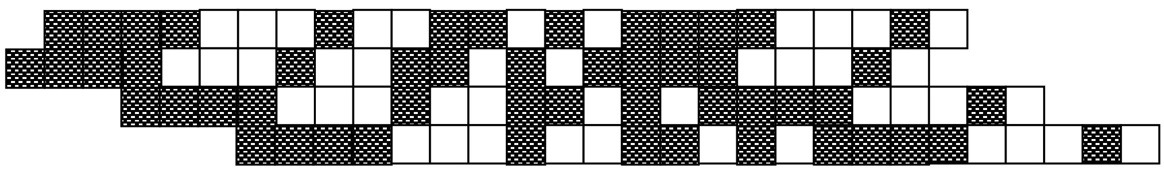
Note that in some of the diagrams in this paper the ends of the strips are beyond the edge of the page. However, the only parts of the strips that are of interest are where all strips contribute to the columns, so just ignore the dangling ends!

# Part I. Two colour problems with four strips.

We can try sliding four strips like this one to get little towers or houses with four sections:



**For example we could slide them around like this:**



To find the "rule of construction" of the sequence we must first remember the simple adding rules for two colours. These are

(a) A colour added to the same colour gives white,
(b) Black and white added will have black as the "sum".

Let us call the sections of each house the floors of the house. So there is a first floor (at ground level), above which is the second floor, then comes the third floor and the fourth floor is at the top.

The rules for building the next house in the sequence seem to be the following:

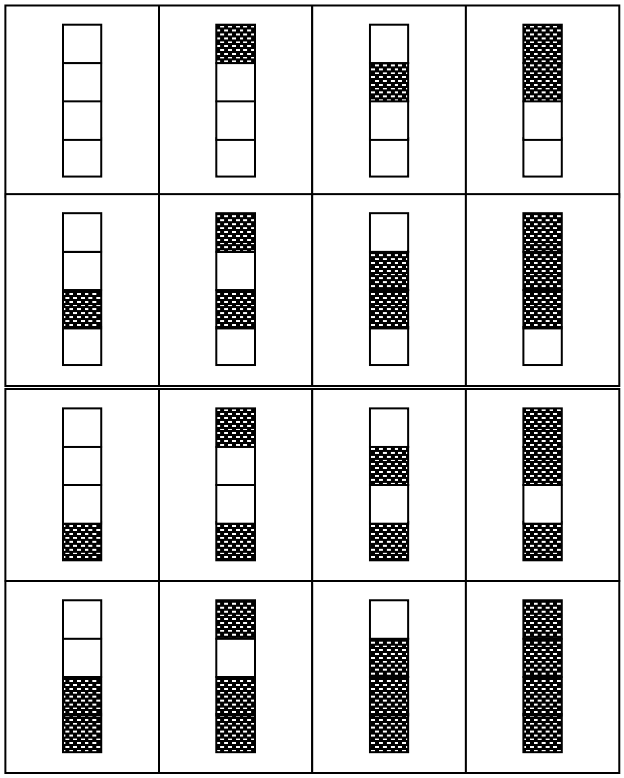
(a) To get the colour of the first floor of the next house, just add all the colours of the house you are looking at except the one of the first floor.
(b) To get the colour of the second floor of the next house, add up all four colours of the house you are looking at.
(c) To get the colour of the third floor of the next house, just add all the colours of the house you are looking at except the colour of the second floor.
(d) The colour of the fourth floor of the next house is always the colour of the third floor of the house you are looking at.

If you want to "add two houses to each other", just add the colours to each other that are at the same levels of the houses.

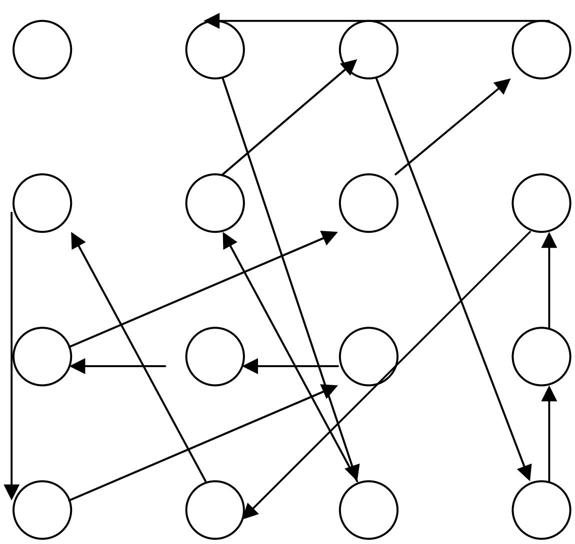
You might find it amusing to check that in any five houses following one another in the sequence, the sum of the first two houses will always give you the fifth house.

Here are sixteen houses arranged in the form of an organized “village”. The all white house, which does not come into our sequence, is placed at the upper left corner



The sequence will allow a “walk around the village” from house to house, visiting every house except the all white one. Here is a “map” of the walk:



You can now organize your houses into sets whose inhabitants can have a picnic party together.

If you want a three-house party, you will already know how to do it, if you have checked on some of the runs of five houses in our sequence. Out of any such run of five, the first and the second will always “add up” to the fifth. You may not be surprised that likewise the second and the fifth will “add up” to the first, and the first and the fifth will “add up” to the second. In other words in a set of three houses for a party, any two of them must “add up” to the third one.

But the villagers wanted more houses to a party, so they thought of having seven houses forming a picnic set. If you want to work out how they might have done it, do not read any further, but try and work out a way on your own. Then you can compare your “solution” with mine, which follows.

(a) Houses painted white on the same floor can make a party. This will give you four ways of making up a picnic set, since there are four floors.
(b) If the colour on two given floors is the same, then such houses can form a picnic set. There are six ways of choosing two floors out of four, so this gives us six more picnic sets.
(c) If the sum of the colours on three given floors is white, they can form a picnic set. There are four ways of choosing three floors out of four, so this gives us four more picnic sets.
(d) If all the colours of the whole house add up to white, then such houses will form another picnic set.

You will see that there are fifteen seven-house picnic sets that we can make up in the above ways.

You might have wondered whether the seven-house sets are regularly distributed over our sequence. The answer to this is in the affirmative. If we imagine the fifteen houses of our sequence placed along a circular pattern, the picnic sets appear in the following order:

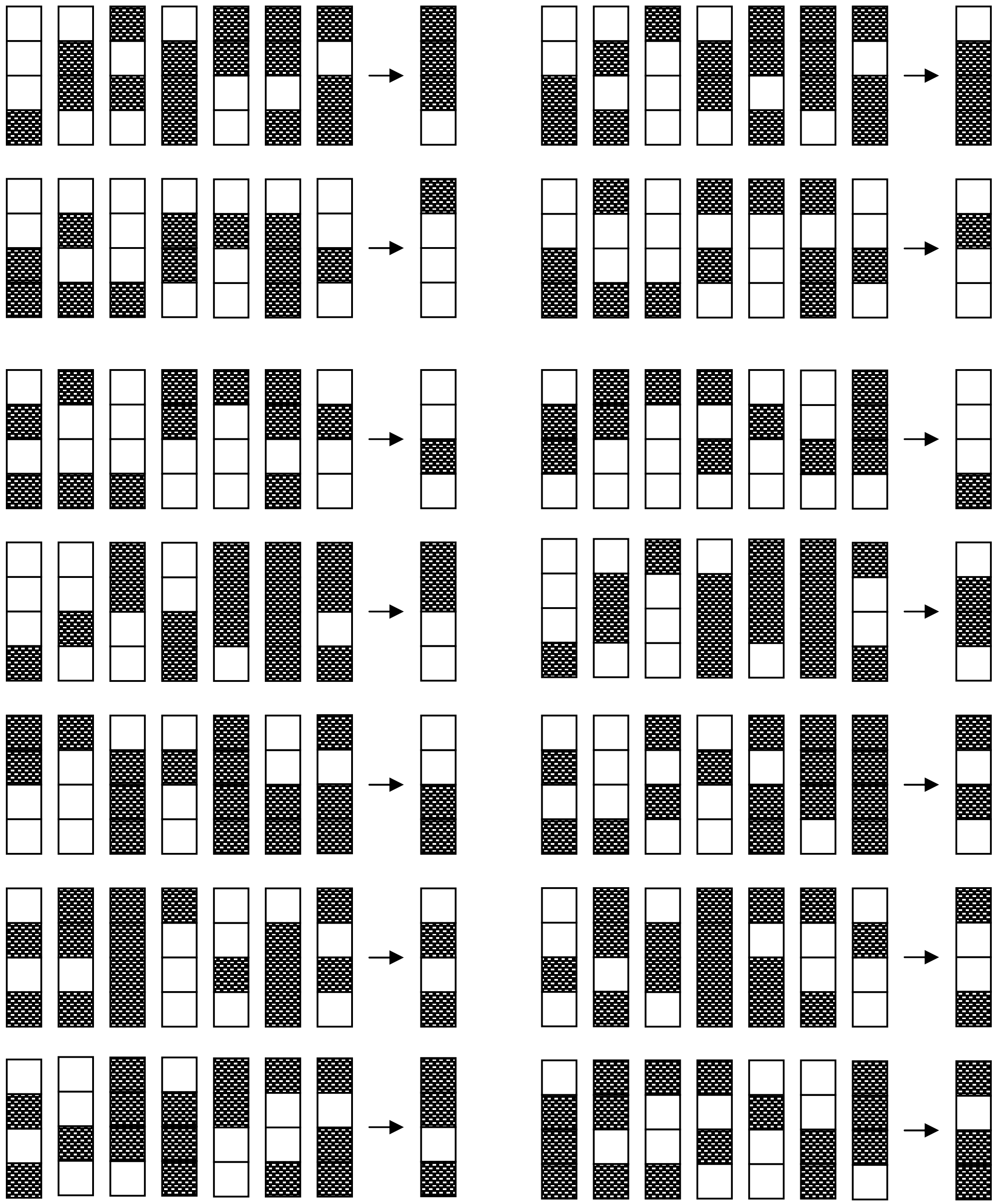
Y Y Y N Y Y N N Y N Y N N N N

where Y means “Yes, the house belongs to the set” and N means “No, the house does not belong to the set”. You can start anywhere in the sequence with three Yes’s and you will get one of the picnic sets if you observe the No’s indicated by the N’s.

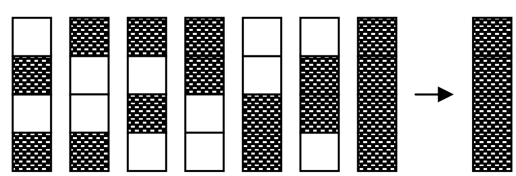
You might like to check on any two of the seven-sets to see which houses they have in common. You will find that there will always be three common houses to any two picnic sets of seven, and what is more, these three houses will form a three-house picnic set!

We can associate a house to each seven-house set in the following way:



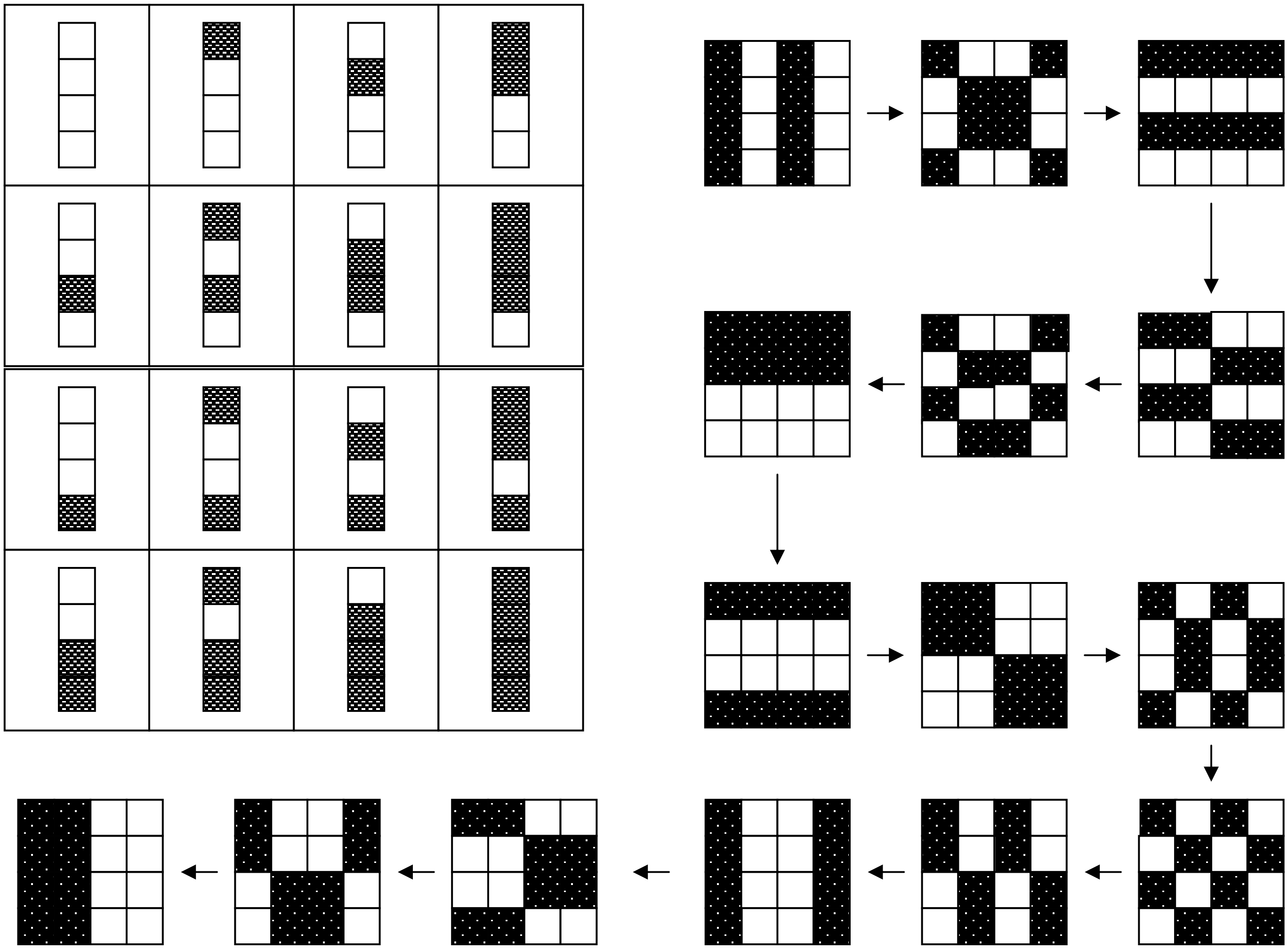
In these sets in any run of four houses the first, the second and the fourth houses always form a three-house-set. Any seven-house-set can be arranged in a cycle  so that this is so!



In the above fifteen seven-house picnic sets, an arrow points each time to the suggested associated house. You can check that if you add those colours for any house within a set that correspond to the black parts of the associated house, you will get white for the sum. So there is a reasonable "rule" for associating a house to a set. You will also find that for any three seven-house-sets that have a three-house-set in common, the associated three houses form a three-house-set!

Here are some maps of the picnic sets as they are situated in the village.



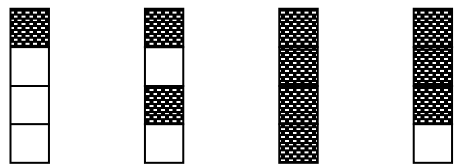
The arrow means applying our sequence rule to get to the next house. You can see that a picnic set is always transformed into another picnic set by following our walk around the village. Applying the rule to the fifteenth set, you again return to the first one. The sets come in the following order, the rows being read from left to right following the order determined by the arrows:

| | | |
|---|---|---|
| **$4^{th}$ floor white** | **Total sum white** | **$2^{nd}$ floor white** |
| **$2^{nd}$ floor = $3^{rd}$ floor** | **Sum of $2^{nd}$ $3^{rd}$ $4^{th}$ white** | **$1^{st}$ floor white** |
| **$1^{st}$ floor = $2^{nd}$ floor** | **$1^{st}$ floor = $3^{rd}$ floor** | **Sum of $1^{st}$ $2^{nd}$ $4^{th}$ white** |
| **$2^{nd}$ floor = $4^{th}$ floor** | **$1^{st}$ floor = $4^{th}$ floor** | **$3^{rd}$ floor = $4^{th}$ floor** |
| **Sum of $1^{st}$ $2^{nd}$ $3^{rd}$ white** | **Sum of $1^{st}$ $3^{rd}$ $4^{th}$ white** | **$3^{rd}$ floor white** |

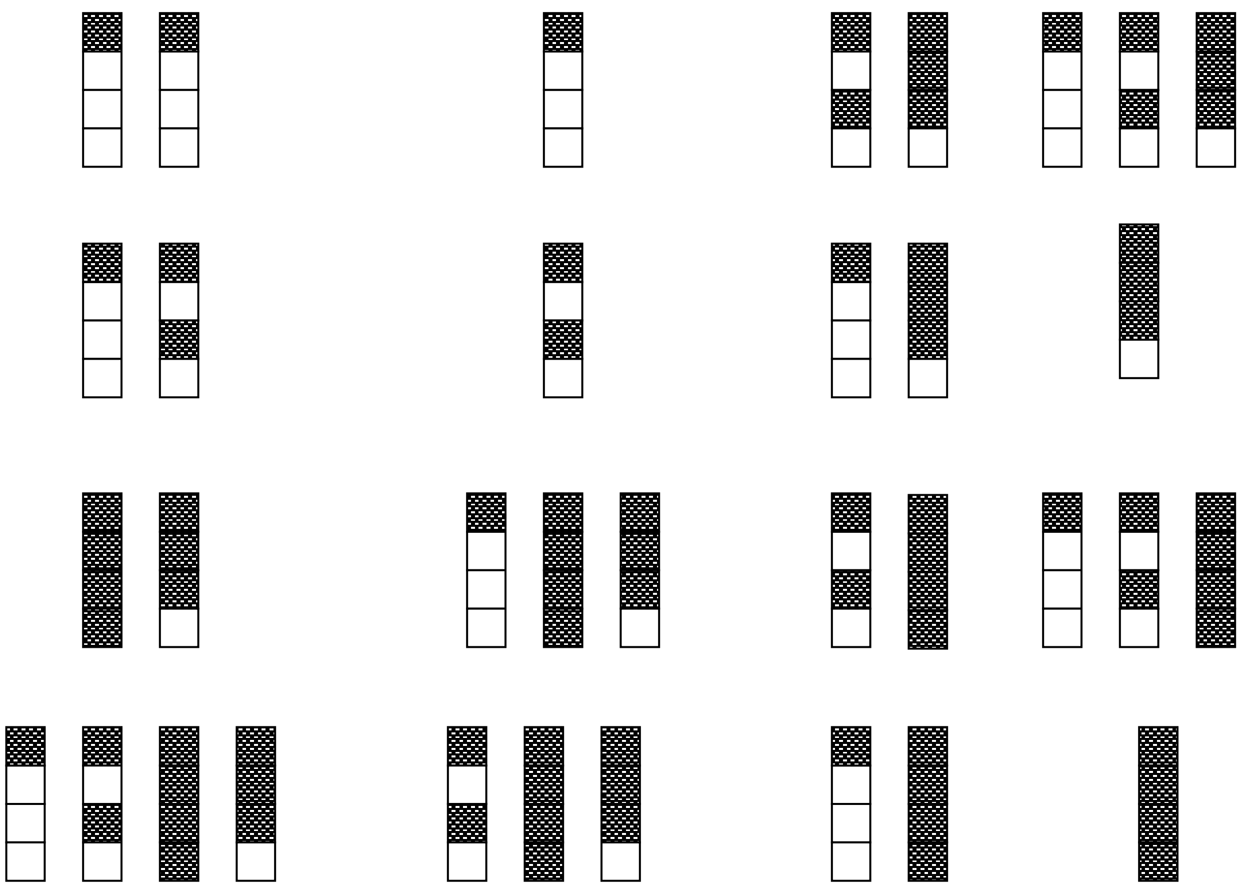
  

Now let us see how we can construct all the houses in the village by combining a certain number of houses with each other. We shall need four houses, but we shall have to exercise some care about how we choose these four houses. First of all, our four houses must not all belong to the same set of seven houses suitable for having picnics! But wait! There is something else to look out for! It is important that no three of our four houses should form a three house picnic set!

Let me choose four houses, which hopefully satisfy the above two conditions:

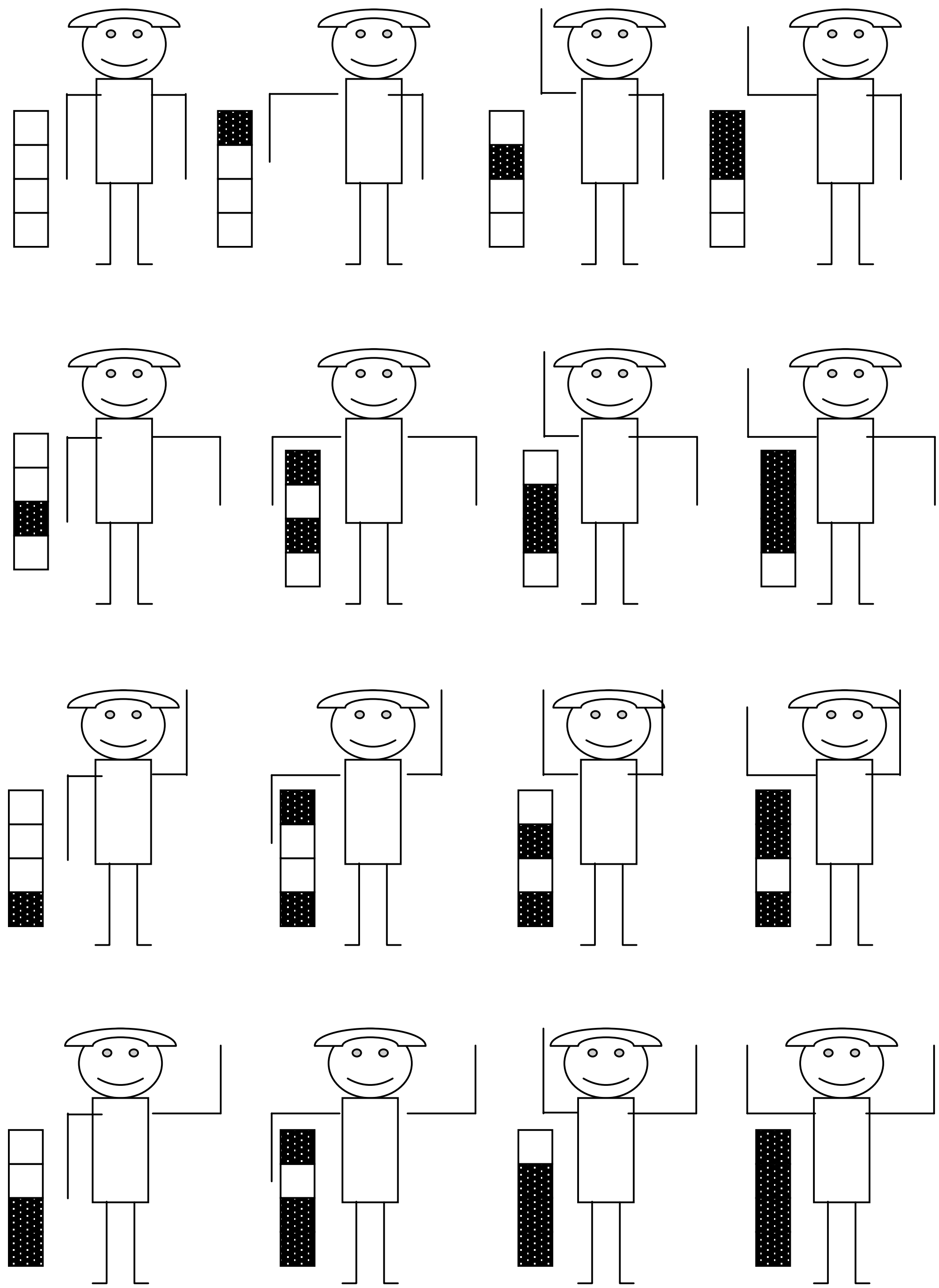


Our village can be reconstructed out of the above four through using the combinations given below:



We could use different positions of a person's arms to represent the sixteen different houses which make up our village. This is how it could be done:

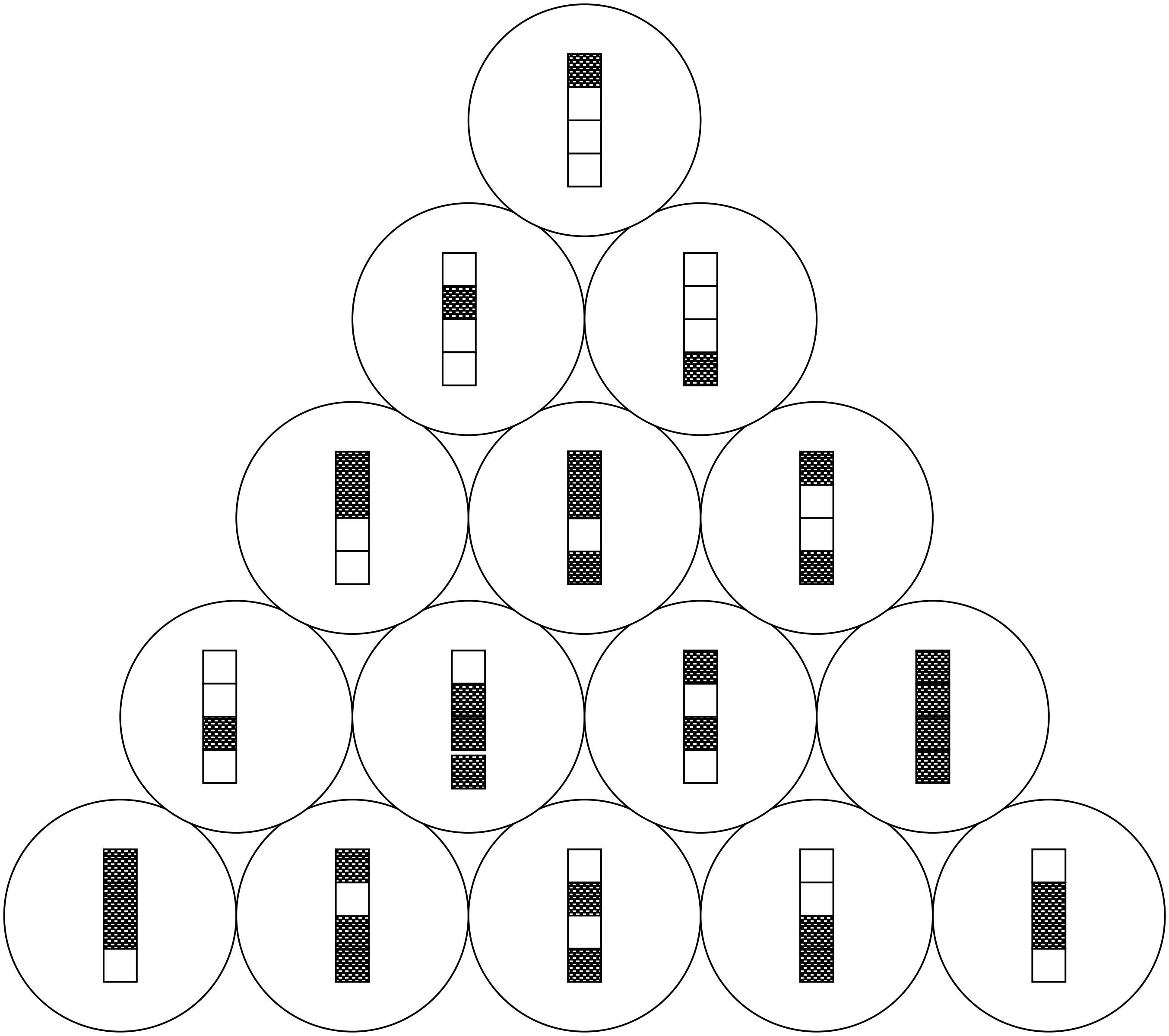


4th floor → right arm bent or straight, 3rd floor → right arm up or down

2nd floor → left arm bent or straight, 1st floor → left arm up or down

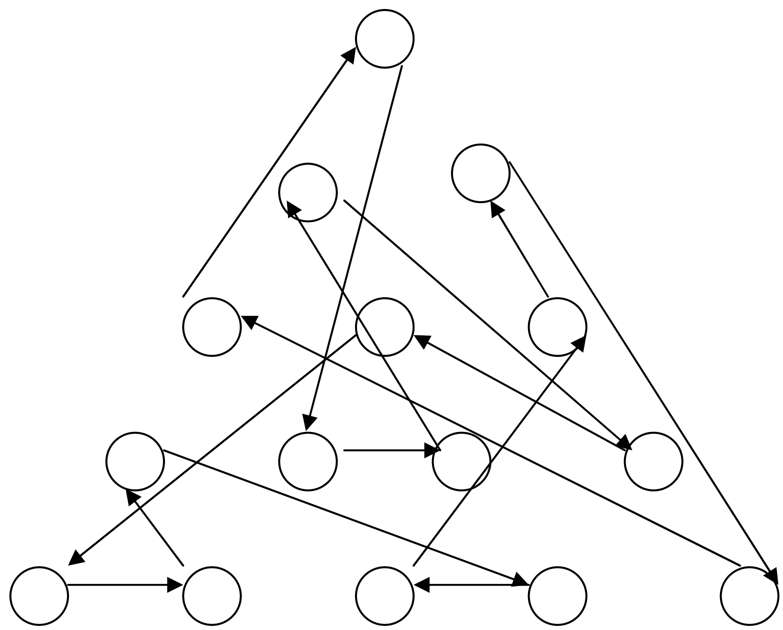
Instead of having rows of houses or people, which you might think is not a very creative way of planning things, we could arrange fifteen houses (or people?) in a triangular pattern like this:



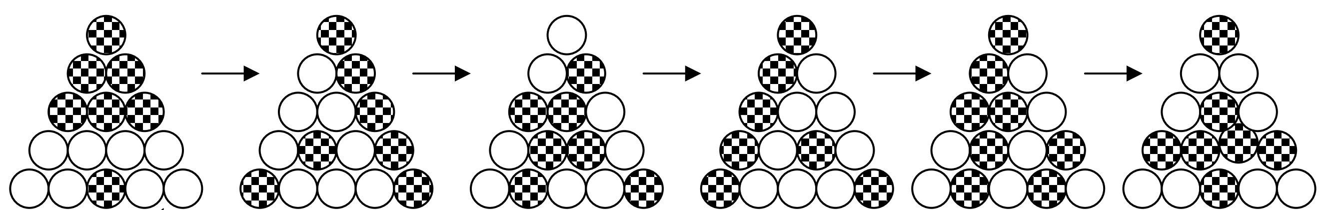
It might be fun to find how the seven-sets as well as the three-sets are placed in the above village. They all make interesting patterns. Try to find them. There are 35 three-sets to find! Not forgetting the 15 seven-sets.

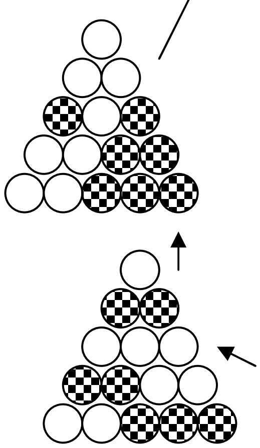
  

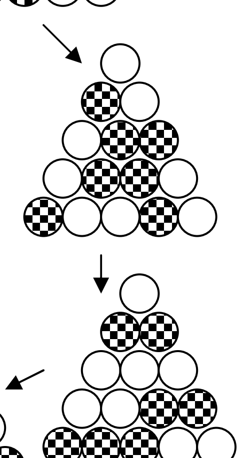
Our sequence will take us on the following walk in the triangular village:



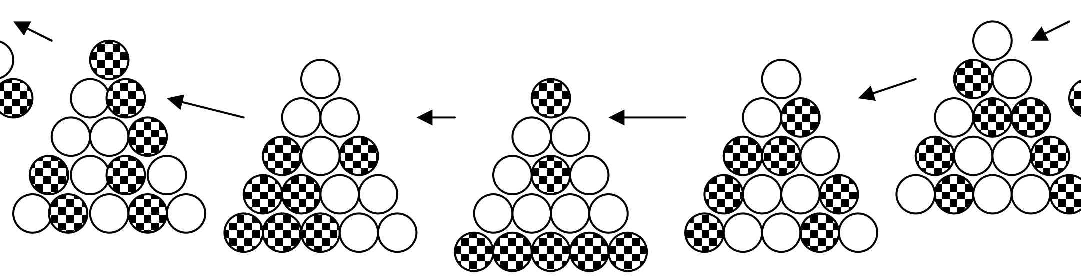
The walk can start anywhere, for example in the seven houses of a seven-set. Carrying out the walk, we shall meet the other fourteen seven-sets in the following order:







You can see five different patterns, each one “pointing” in three different directions, thus yielding fifteen patterns. You might like to check that in any run of five sets the first the fourth and the fifth have a three house set in common. Try to find some other features of the sequence.



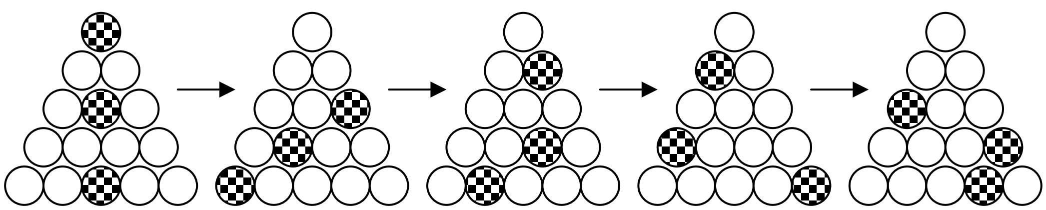
If you are looking for a really challenging problem, try this one:

Place the above fifteen sets of seven in three rows of five, which will therefore give you five columns. Arrange them in such a way that the three sets in each column have a three house set in common, and also so that the first three sets as well as the last three sets in each row have a three house set in common. This should keep you from being bored on a stormy afternoon!

Let me introduce an operation which you can perform on two of the sets of seven. I could call the operation “joining”. The houses in the “joint set” of two sets consist of those houses which either are in both sets or are absent from both sets.

Try to “join” two sets on either side of an arrow. If you “jump over” two sets in the sense opposite to that of the arrows, miraculously you will find the “joint set” of your two “neighbouring” sets. If you pick two sets separated by one other set in the sequence, you can count seven arrows forward from the second of your two sets, and you will find the joint set. Try to find some more “rules” like that.

You might want to see which three-house-sets we get as we progress along our walk. Here is one possibility (after the fifth set we come back to the first):



There are more such “runs ”. Try to find some of the others. Can you find any runs of fifteen in which you meet fifteen different sets of three? You might even find two if you look hard enough!

Of course the above run is really a run of fifteen, because although you come back to the same set of three after five applications of the sequence rule, each house comes back to a different house.

For any feedback, do not hesitate to contact me at zoltan@zoltandienes.com

# Part II. Three-colour problems with two strips.

## 1. How to add colours for making strips.

Let me start by using three colours, for example yellow, blue and red. We can get on to more colours later! You could equally well use three shapes such as circles, squares and triangles, or three kinds of beings such as for example boys, girls and cats.  I shall stick to the three colours.

Let me suggest a way of "adding" the colours.

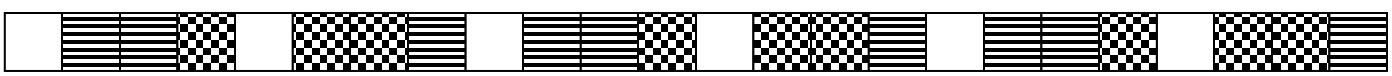
(a) Adding yellow to any colour will not change that colour.
(b) Adding two identical non-yellow colours to each other  will result in the other non-yellow colour.
(c) Adding blue and red will always result in yellow.

Now let us make a strip, using our colours. Cut a fairly long strip, about 2 cm wide, and  divide it into little spaces 2 cm long. Make sure your strip has at least  twenty little squares.

Colour the first two little squares, using your colours. They can be both the same colour or not, but do not make them both yellow.

For colouring your third little square, just add the colours of the two squares you have just coloured. For colouring the fourth little square, add the colours of the second and the third. The colour of the fifth one will be the "sum" of the colour of  the third one and of the fourth one. Go on like this, always adding the last two colours to get the next colour. Go on till you have coloured the whole strip.

You will find that whichever way you start, as long as you do not start with two yellow ones, you will get a repeating cycle of eight like this:



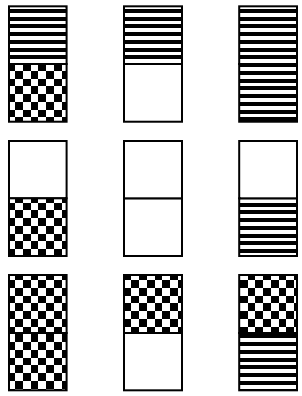
where blank means yellow, level stripes mean blue (a lake?) and square tiling means red (a roof?)

Make two strips like the above, making sure that your little squares are all equal in size.

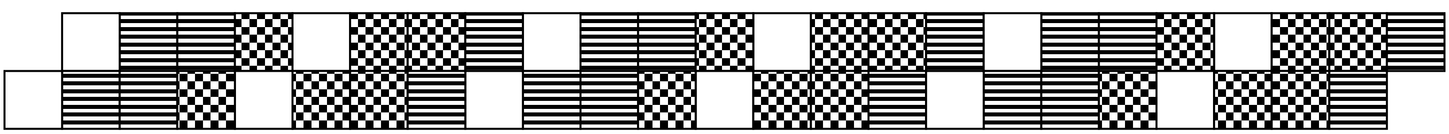
## 2. Making the "village".

Now take six yellow, six blue and six red interlocking cubes and make a little "village", each "house" in the "village" being one cube placed on top of the other. All the houses should be different and should be arranged as shown below

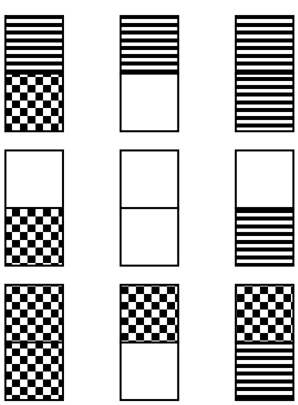


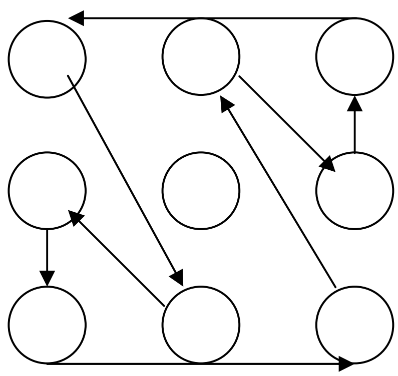
## 3. Sliding the strips for making the "walks".

Use your strips now to make up rules about going for a walk in your village. For example, supposing you place your strips next to each other (one "below" the other) so that all the colours match and then you slide the lower strip one "notch" to the left. You will get the following effect:



which will give you the following walk:



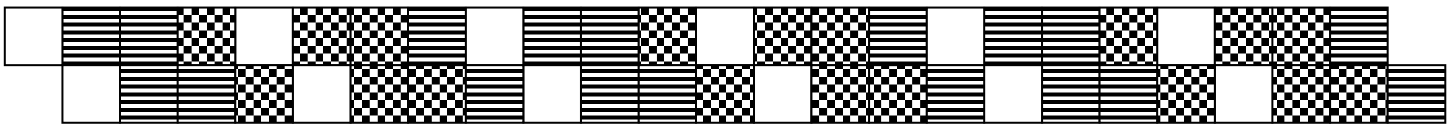


Our “walk” follows quite a simple rule as we go from house to house. It seems to be this:

The lower section of the house where we are will have the colour of the upper section of the house where we are going next.
The sum of the lower colour and the upper colour of the house where we are will be the colour of the lower section of the house where we ae going next.

If we slide the lower strip to the right instead of to the left, we shall get the following set of instructions for our walk:



The upper section of the house where we are now will have the colour of the lower section of the house where we are going next.
Here the sum of the upper and the lower colours will give us the colour of the upper section of the house where we are going next.

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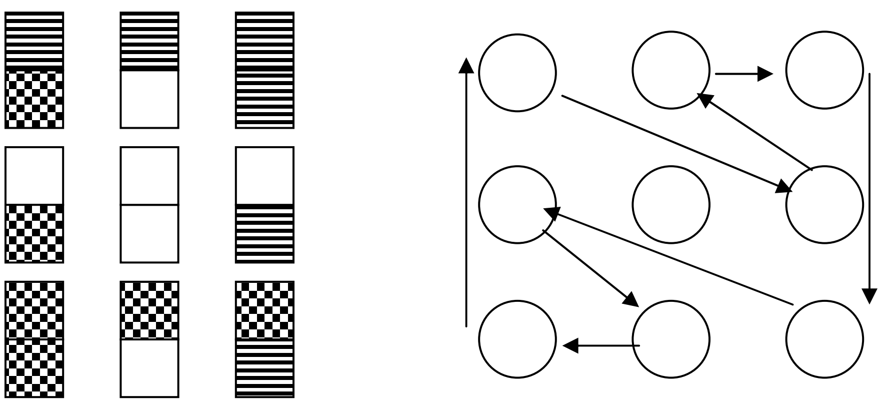
Here is the map of the walk.



In both walks we visit every house in the village except the middle one, which is all yellow.

Of course we could have read the strips from right to left instead of from left to right. We would have obtained the “inverses” of these walks. An “inverse walk” simply undoes what the other walk has done, of which it is the “inverse”

Then it would be possible to move the lower strip two notches to the right or to the left, or else three notches to the right or to the left. It would not be much use moving the strip four notches, as we would not then get a walk at all. So if we count the inverse walks as distinct, we can arrange twelve different walks during each of which all houses are visited except the middle one.

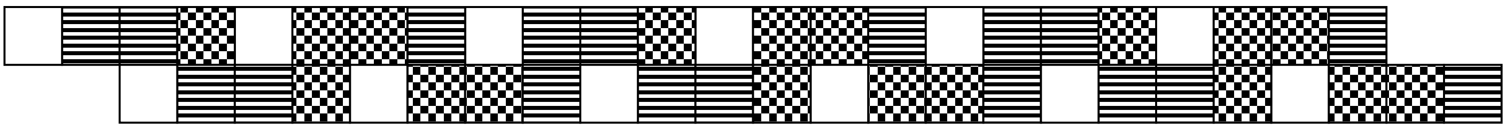
  

# 4. The "swapping" rule and six-house walks.

Let me introduce another interesting little operation, which I shall call SWAP. The rules for "SWAPPING" are as follows:

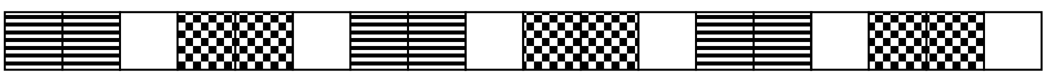
SWAP yellow = yellow, SWAP blue = red, SWAP red = blue

It will be useful in describing some of the walk-rules that arise if we slide our strips by more than one notch. Here is what happens if we slide the lower strip to the right by two notches

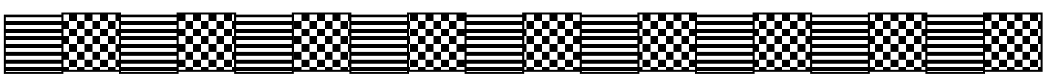


If we swap the lower as well as the upper colour of a house and then add the swapped colours, we obtain the colour of the upper section of the next house.
If we add the upper colour to the swapped lower colour of a house, we obtain the colour of the lower section of the next house
If we move to the left instead of to the right, we get a reversal of the rules as before.
You should now be able to find the rules for moving the lower strip three notches each way, as well as draw the maps of the corresponding walks.

I am sure you must have wondered if there were any strips you could slide so as to visit less than eight houses in the village. I thought I would show you some strips that will give you rules for visiting six houses. Here are two such strips, an A-strip and a B-strip:



A



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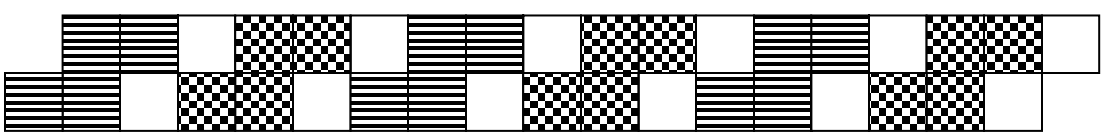
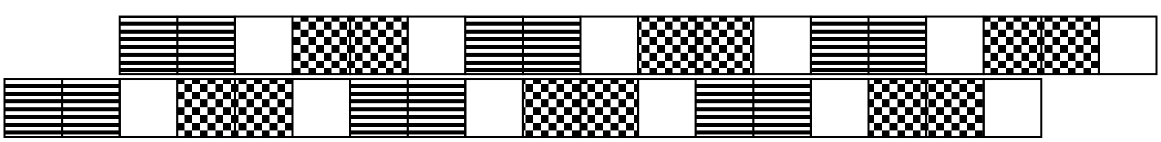
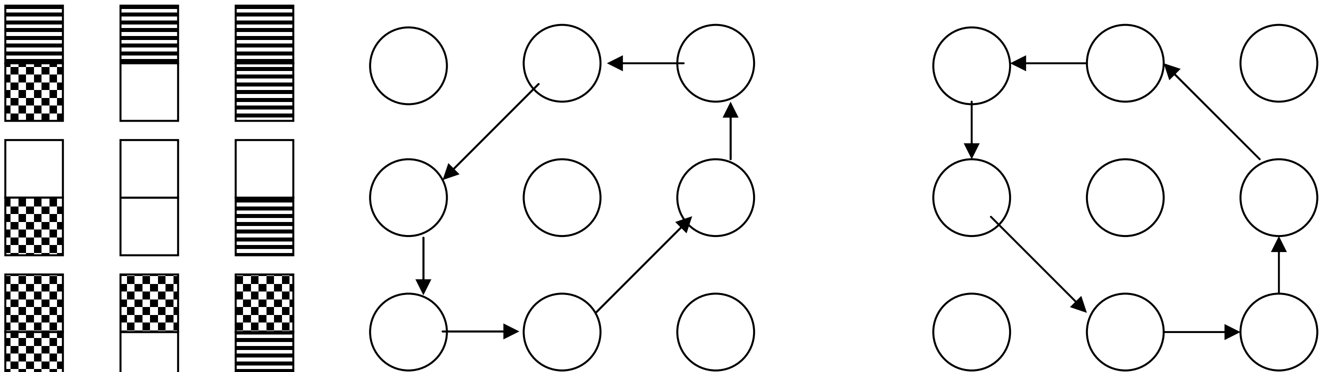
You can slide two A-strips along side each other, or you could have an A-strip sliding along with a B-strip. For example you could have



Or else



which would give you the walks:



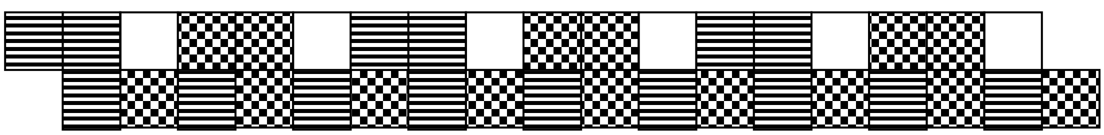
Sliding the strips the other way we shall get the same two walks but backwards.

But what about the rules for each walk?

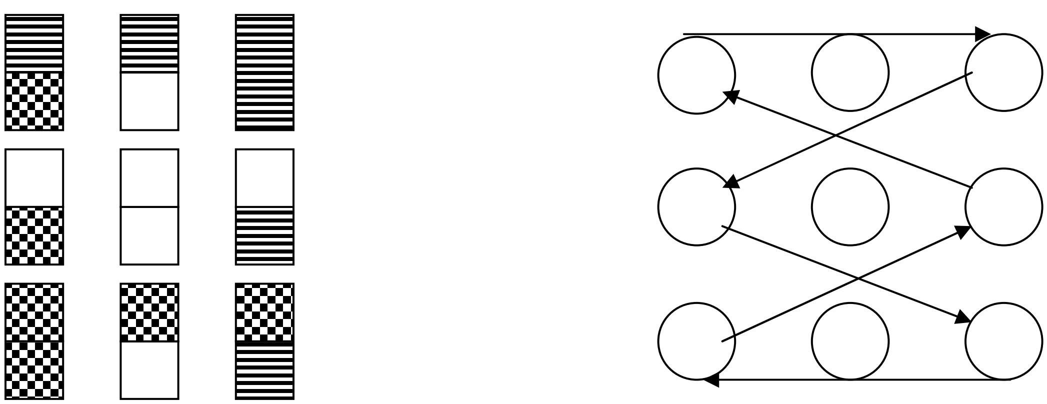
For our first walk each lower colour will give us the upper colour of the next house; The upper colour swapped, added to the lower colour will give the lower colour of the next house. This also works if we go back and forth between the “unvisited” houses!

For the second walk the sum of the lower colour and the upper colour will give the upper colour of the next house, and the swapped upper colour will give the lower colour of the next house. This also works in walking between the “unvisited” houses! Using two B-strips will generate more “boring” walks.

Now let us see what happens if we use an A-strip with a B-strip. For example try this:

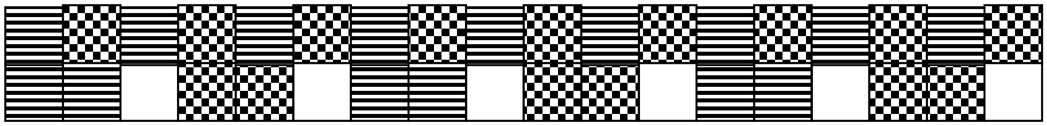


It will give this rather more energetic walk:

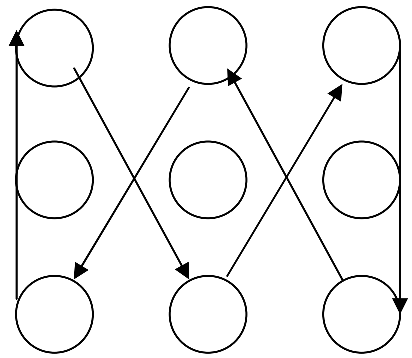


Or we could try this:



Which will give this walk:

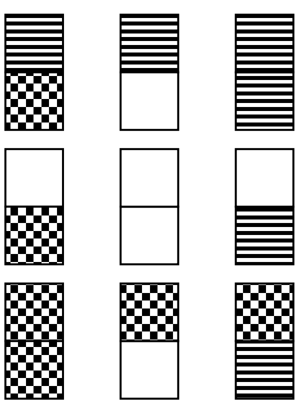


The rules for the first walk seem to be these:

The swapped lower colour will give the lower colour of the next house and the swapped upper colour added to the lower colour will give the upper colour of the next house.

In the second walk it is the upper sections that alternate between red and blue, and the lower colour of the next house is given by the sum of the swapped colours of the previous house. These rules also work for walking back and forth between the "unvisited" houses.

## 5. Houses whose inhabitants can party together.

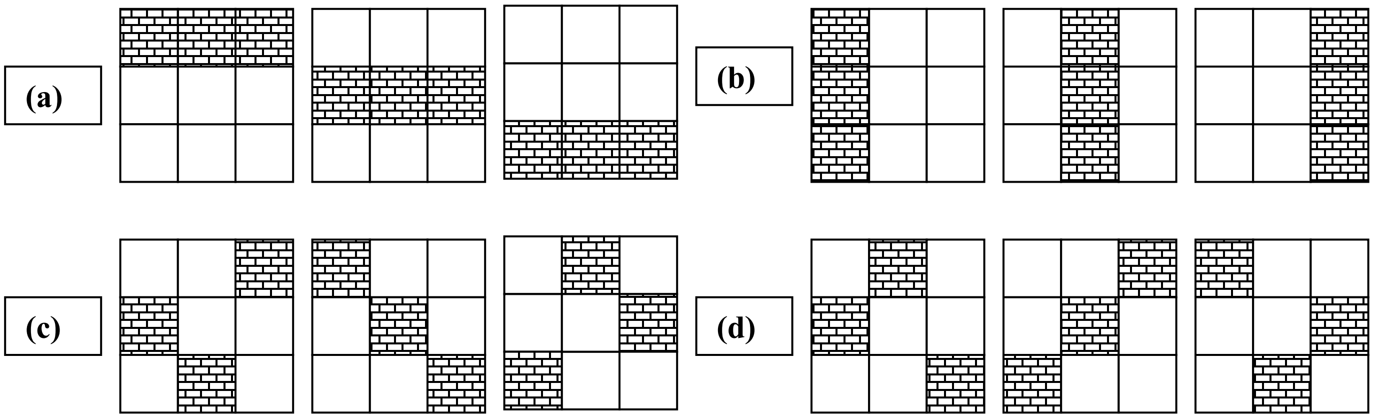


The inhabitants of the village like to organize parties at which people from three different houses participate. They decide on some rules as to which three houses can form the basis of a good party. These are the rules they have thought of:

(a) The upper section of all three houses have the same colour.
(b) The lower section of all three houses have the same colour.
(c) The sum of the upper and lower colours is the same in all three houses
(d) The sum of the swapped upper colour and the lower colour is the same in all the three houses.

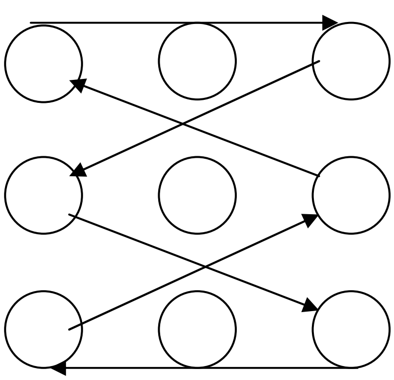
  

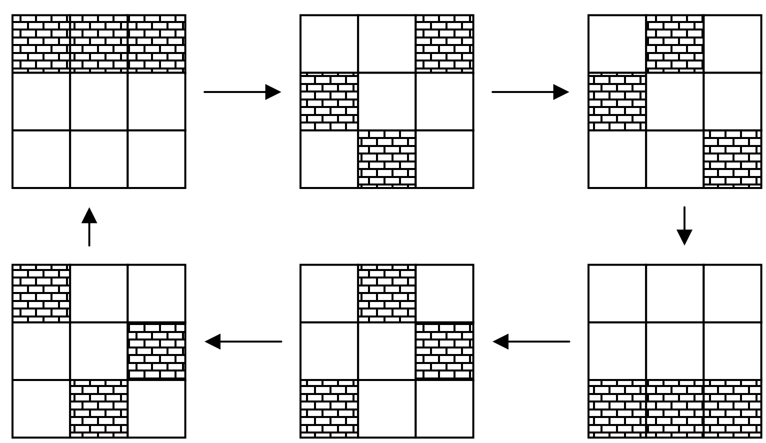
Since there are three colours, the villagers can have parties in twelve different ways. Here are the different ways of being allowed to select houses for a party:




One day all the villagers decide to move house. They select one of the walks, either an eight house walk or a six house walk and each family moves to the house where the selected walk leads them from the house they occupy at present. The family living in the house in the middle of the village of course does not move, as no walk ever takes in that house! When they next want to organize parties, they realize that exactly the same families as before can go together to the same party! Try to check in some cases that this is true. Let me try just one, in which the upper middle and the lower middle also exchange with each other following the moving rule!

For example





# 6. Have we made a geometry?

You might have noticed that two sets of three houses determined by the same rule never have a house in common, but two sets determined by two different rules always have a house in common. So let us make a big leap into Geometry!

Each house is going to represent what we in Geometry call a point. Each set of three houses that are admissible for running a party is going to represent what in Geometry we call a line. A rule for determining a line will be called a direction. Two lines determined by the same rule will be called parallel as they have the same direction.

According to the above any line consists of three points and has a direction.

Also any point forms part of four different lines, as is easy to verify.

Some crazy mathematicians have called a direction a "point at infinity", as by doing so they can say that every line has four points, as well as every point has four lines.
They can also say that two points determine a line and two lines determine a point.
All the "points at infinity" (directions for you and me) make up the "line at infinity" (the set of all the directions, for you and me)

Actually there are 48 ways of organizing walks in such a way that lines will always be turned into lines. For this reason such transformations are called linear ("line-like").

If you want to know more about these 48 transformations, look in my book entitled

'*I Will Tell You Algebra Stories You Have Never Heard Before*'

published by UPFRONT Press, Leicester, England, ISBN: 1-84426-191-3.

It can be purchased via my web site at http://www.zoltandienes.com

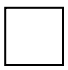
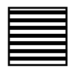
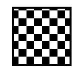
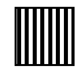
Or you can e-mail me at zoltan@zoltandienes.com

# PART III.
# Two strips and four colours.

I am sure every reader of Part II will have wondered what would happen if we had more than three colours. What kind of adding rules might be suitable and what would happen to the Swapping operation? In this section I will try to show what can be done with four colours. For the moment I will leave to you the extension to more than four colours as a fun problem.

Let us use yellow, blue, red and green. I shall denote these colours by means of the following patterns:

| | | | |
|---|---|---|---|
| **Yellow (desert)**  | **Blue (lake)**  | **Red (roof)**  | **Green (forest)**  |

Let me suggest the following adding rules for our four colours. If you can think of any better ones, be my guest!

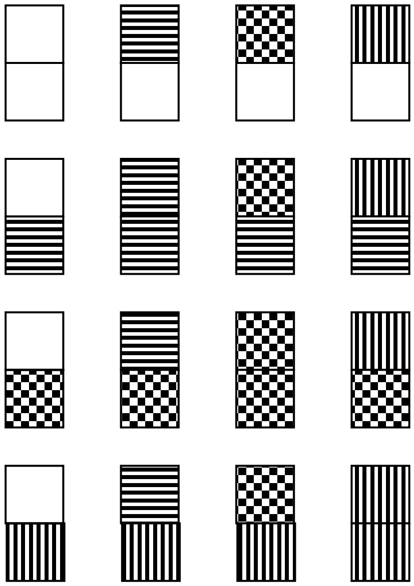
(a) Adding yellow to any colour leaves that colour unaltered.
(b) Any colour, added to the same colour, results in yellow.
(c) The sum of any two different non-yellow colours is the third non-yellow colour

Then instead of the swapping operation, let me suggest two operations

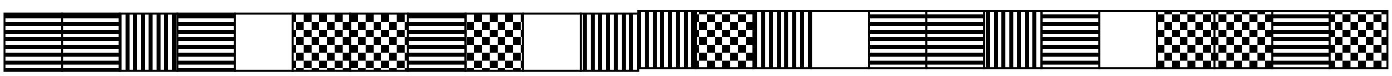
(a) REDDING a yellow is yellow. Otherwise REDDING changes blue into red, red into green and green into blue.
(b) GREENING a yellow is yellow. Otherwise GREENING changes blue into green, green into red and red into blue.

Now let us make our first strip. Have at your disposal a good number of coloured interlocking cubes, using the colours above, unless you have some particular colour preferences. Put down two cubes, but not two yellow ones. Then find the sum of the first colour redded with the second colour. Find a cube of this sum colour, which will be your third cube. Now red the second colour and add this redded colour to the third colour. This will be your fourth colour. Carry on like this, always adding a redded colour to the next colour to obtain the colour that comes after it, until you have a long strip of thirty or more interlocking cubes. Then make another strip just like the one you have just made. These will be your sliding strips for determining the walks through the village. But, of course, we have not yet built the village. Here is a good plan, each house in the village consisting of a lower cube and an upper cube. You might find a better way of arranging the houses!

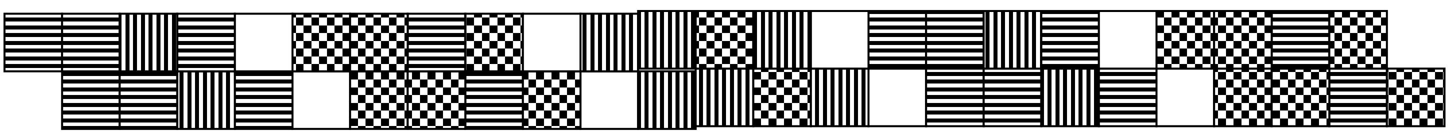
  



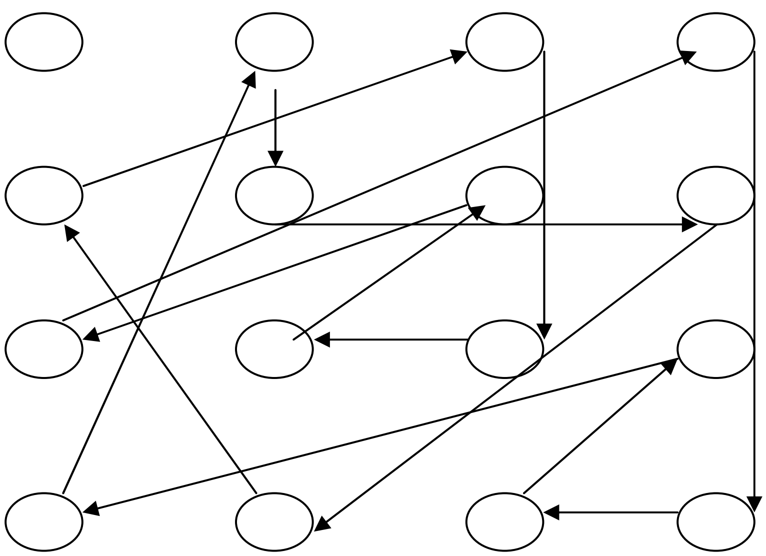
Your strips will consist of runs of fifteen, I believe, and will look something like this:



With two such strips you can do a one notch slide to the left with the upper strip:

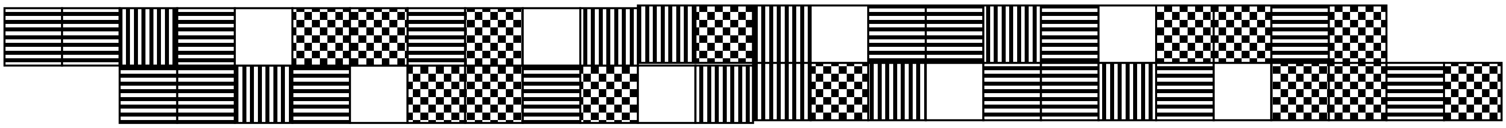


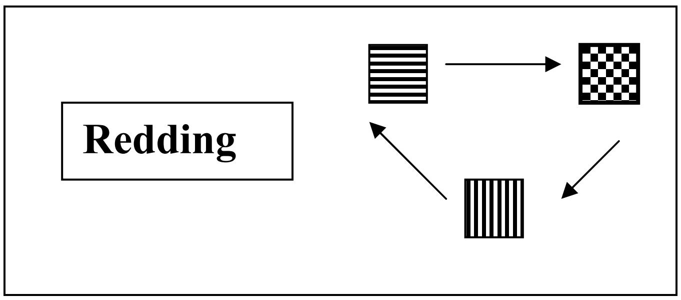
which lets you visit fifteen of the sixteen houses of the village, the walk looking like this:.

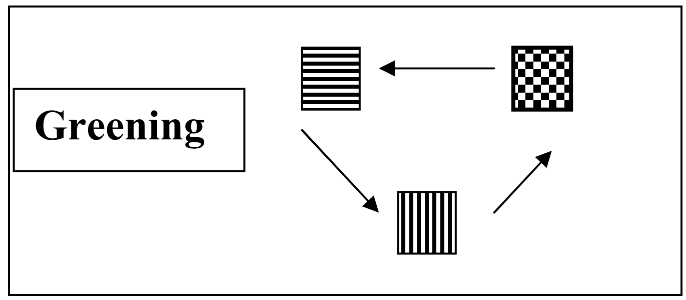


Let us see what happens if we slide the upper strip two notches to the left. The clue to this walk is given by:



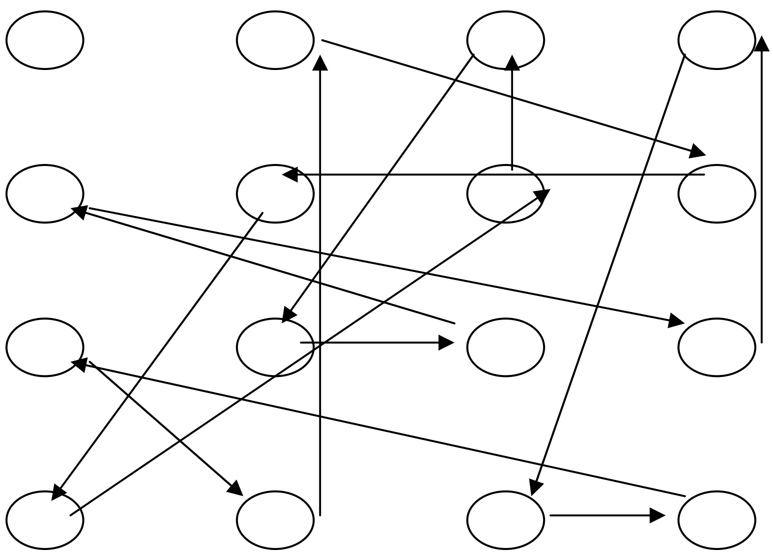



The upper colour added to the redded lower colour in any house will give you the lower colour of the next house.

The greened lower colour added to the greened upper colour of any house will give the upper colour of the next house.

The walk will look like this:



Maybe you can find a better way of arranging the houses, so that the walks do not seem so complicated!

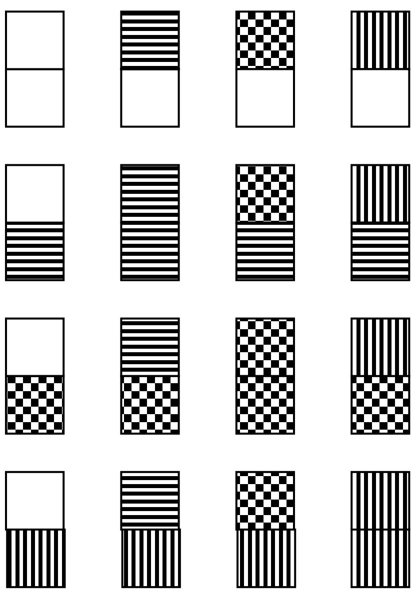
Now let us see how the inhabitants of this somewhat larger village would have organized the rules about which houses could "belong together" from the point of view of going to a party.

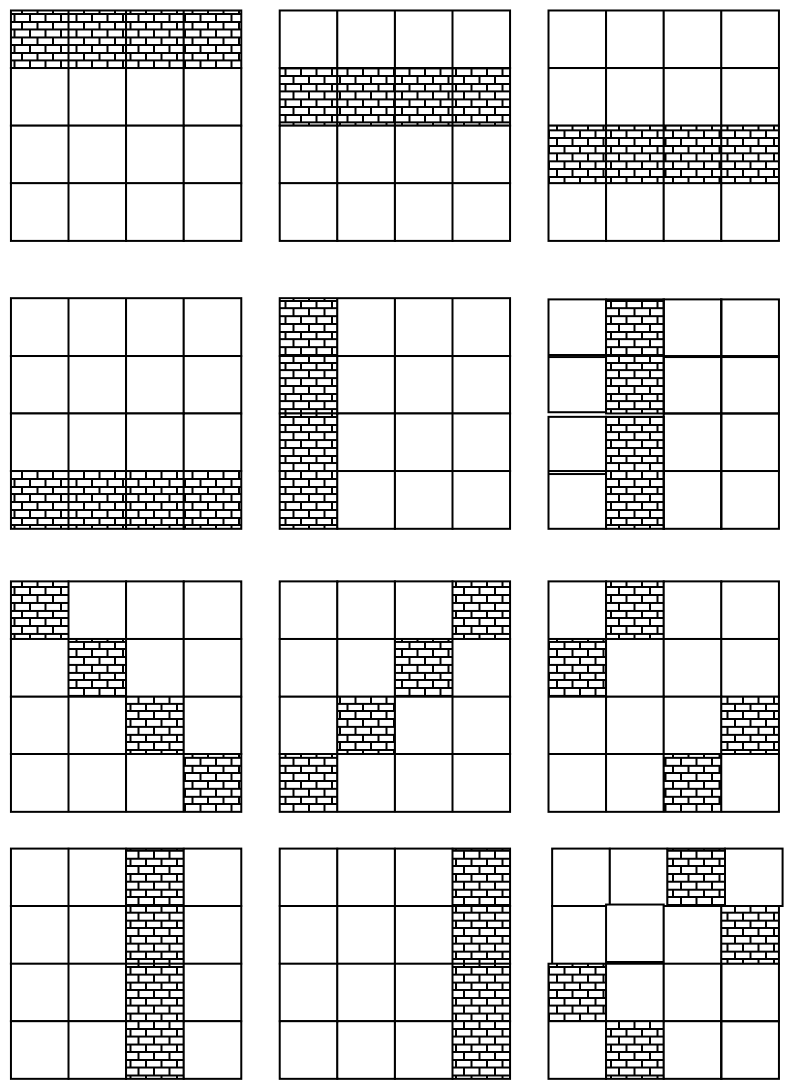
  

I suggest that we should use the inhabitants of four houses for the parties, and the following rules might be worth looking at:

(a) The lower colour of every house in the set is the same.
(b) The upper colour of every house in the set is the same,
(c) The sum of the lower and upper colours is the same for all houses in the set.
(d) The sum of the lower colour and the redded upper colour is the same for each house in the set.
(e) The sum of the lower colour and the greened upper colour is the same for each house in the set.

Since there are four colours, this gives us twenty different ways of putting houses together for a party.

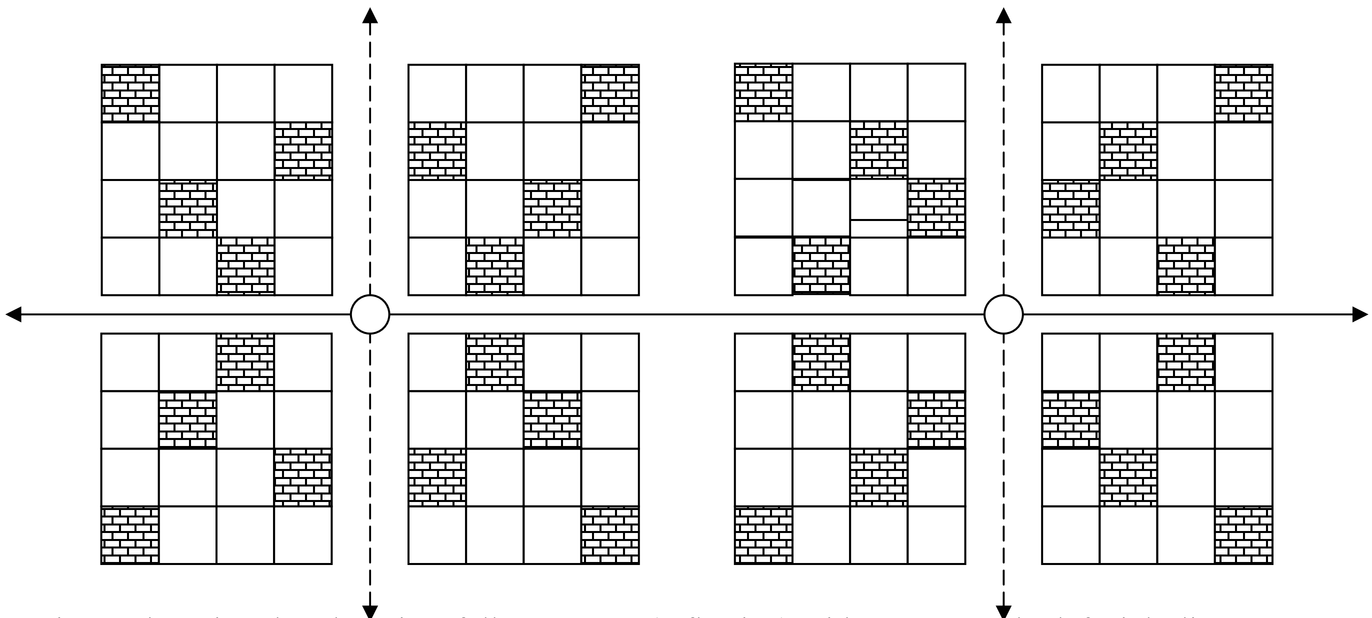




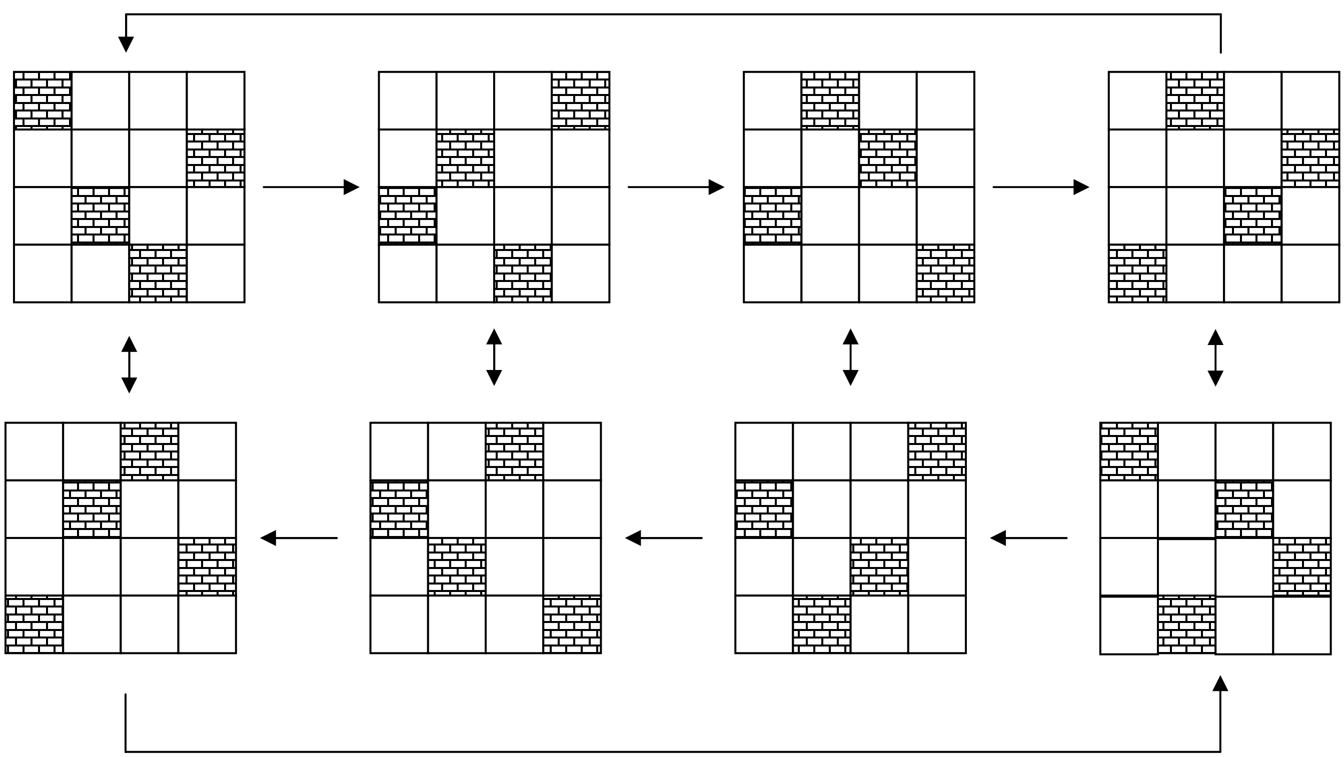
Here are the arrangements of the Houses for the rules (a), (b), (c).

Here are the arrangements arising out of the rules (d) and (e):



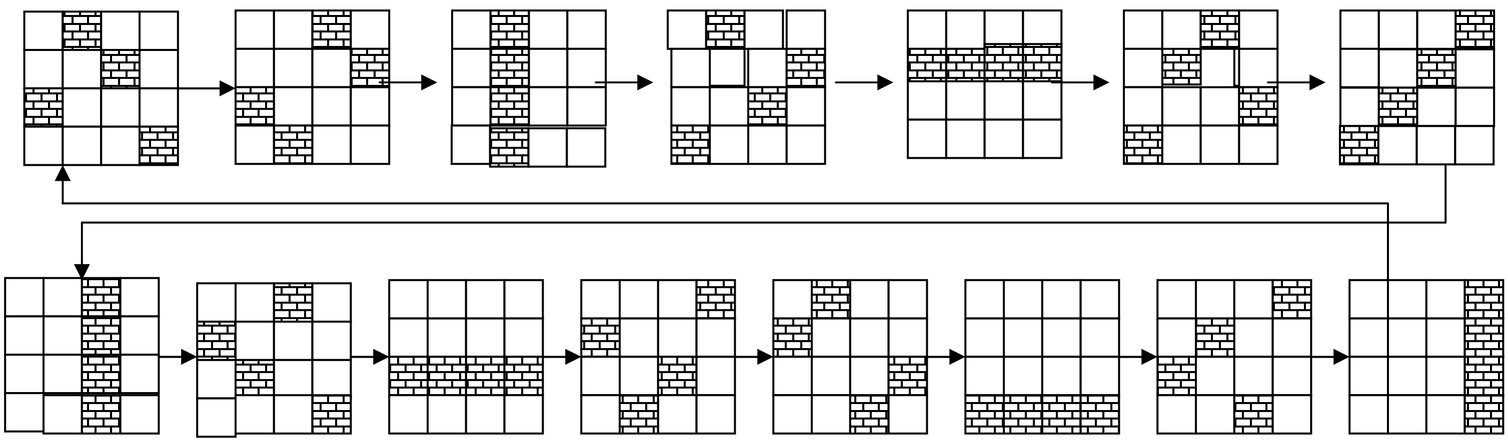
It is worth noting that there is a full symmetry (reflection) with respect to the left-right line, as well as reflection of the neighbouring figures with respect to the dotted lines. There is also a point symmetry (or half turn) about the little white circles. Bringing in two arrangements from each side, we can also place them a "quarter turn apart", as shown below:



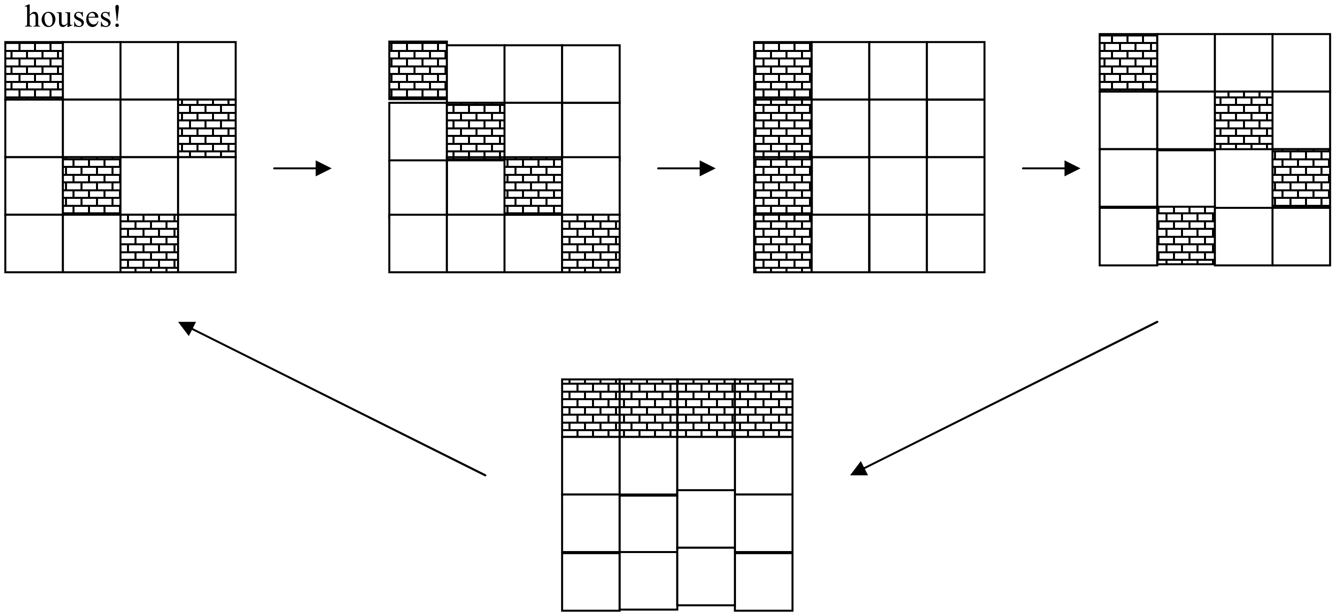
The single arrow in the second representation of the eight sets of houses can be interpreted as a clockwise quarter turn of the village about its centre. The double arrow then means a reflection about the line half way between the upper row and the lower row. So the passage from any one of the eight diagrams to any one of the others can be carried out by using one of isometries of the square, namely either by a quarter turn, by a half turn or by a reflection about one of the four axes of symmetry.

Let us see if we can play the "removal game" with these sets of houses.
We could consider the rules for our second walk for testing whether the walks are "linear transformations".

If you want to see what happens in a fifteen move catastrophe, you could check whether I have these right. Each permissible set of houses is transformed by the walk into another permissible set of houses like this:



If you include the all yellow house, you will get this sequence which looks as though it is a five sequence, but it is not. After five moves people come back to the same set but to different houses!



It seems that we have made a geometry in which there are twenty lines, each line having four points and a direction. Lines determined by the same rule have no points in common, but they have the same direction, so we can call them parallel. I will leave it to you to verify that each point belongs to five lines.

Each walk that we have made (apart from the ones that are left to be made!) will transform our set of sixteen points in such way that lines will be transformed into lines, so the transformations are linear (line-preserving)

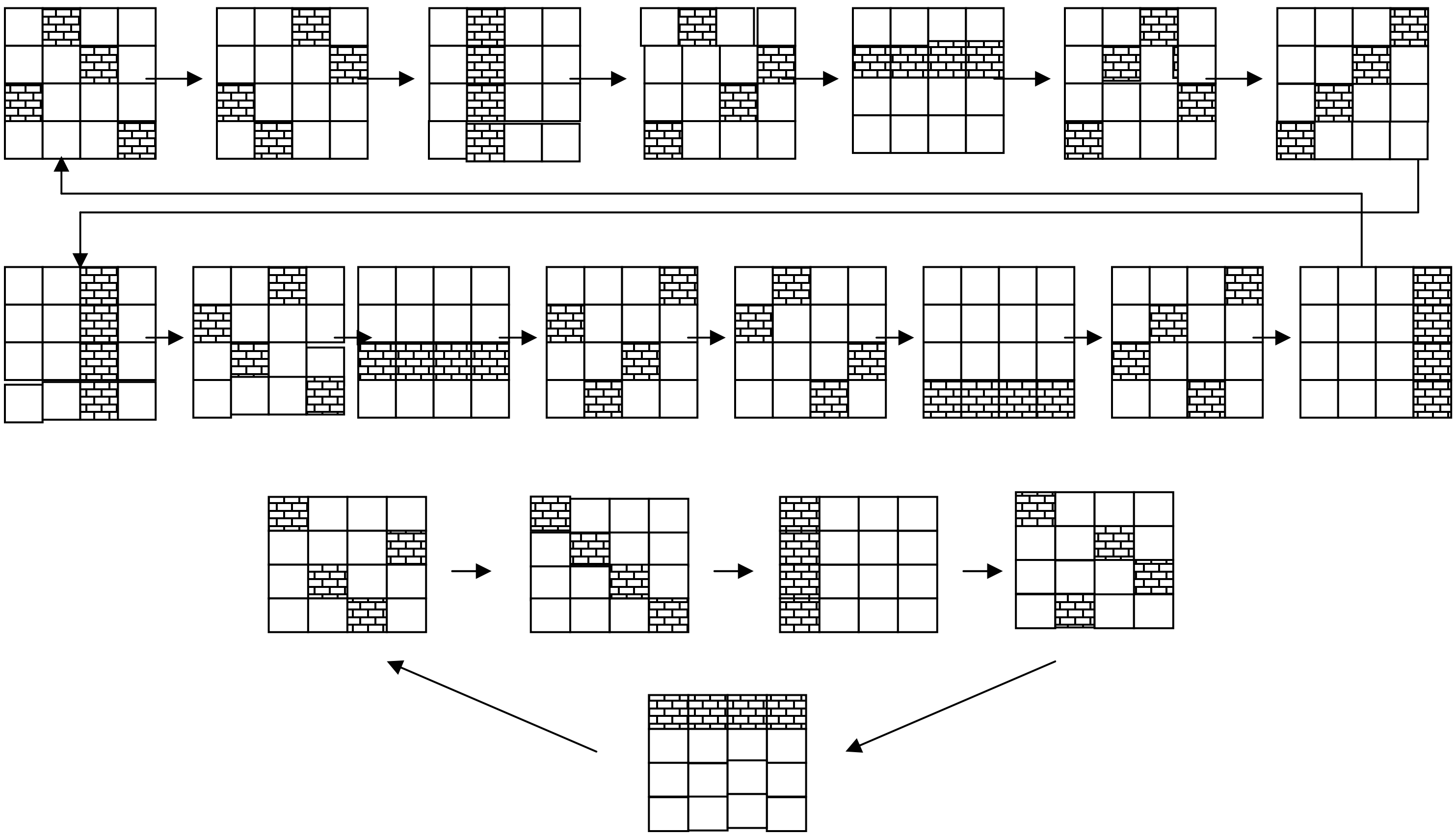
The five directions can be called the "line at infinity", each of the directions being a "point at infinity". In this way of speaking, we have 21 points and 21 lines, and any two lines determine a point and any two points determine a line, not forgetting that a line or a point might happen to be "at infinity"

I will leave it to you to work out the rules for a 25 point geometry in which there would be six directions (points at infinity) and therefore 31 points and 31 lines, counting in the line at infinity. You will need five colours and some intuitive imagination for making up the adding rules and the "swapping rules" for your five colours. Have fun!

# PART IV. Further investigation of four colours

No, I will not give away the rules for the five colour game, but instead I will suggest some more interesting things you can do with the four colours! Below you will find the twenty sets of houses whose inhabitants can go together to a party, linked by our second walk in the first case and linked by three applications of the second walk in the second case..



Let us see how we can recognize the "direction of a line", or speaking the language of the game, how we can recognize the rule determining a set of houses. The position of two of the houses in the village should already determine the rule, and so determine where the other two houses are whose inhabitants can go to a party with them. Here are some fairly obvious rules:

(a) If two houses are in the same row, then the other two are in that same row, All the lower sections will have the same colour.
(b) If two houses are in the same column, then the other two are in that same column. All the upper sections will have the same colour.
(c) If two houses are in the same diagonal, then the other two will be in the same diagonal. In one diagonal each house will only have one colour, in the other they will be either yellow-green or blue-red.
(d) If two houses are in the same "diagonal rectangle", the other two will be in the same "diagonal rectangle". That means that in one rectangle each house will be either yellow-red or blue-green, and in the other one they will be either red-green or yellow-blue.

So here are the rules:

(a) Two houses in the same row: complete the row.

(b) Two houses in the same column: complete the column.

(c) Two houses in the same diagonal: complete the diagonal.

(d) Two houses in the same “diagonal rectangle”: complete the “diagonal rectangle”

We shall have to do some thinking before we can decide on the rules for the remaining eight sets of houses.

First let us note that a “corner house” is always included in each one of these eight sets. There are two sets having the same corner house in them, and they can be obtained from each other by a reflection about the diagonal of the village through the corner house in question. Now let us look at the house which is nearest to a corner house we are looking at. It will be found by performing a “knight’s move” from the corner house. Sometimes this move will be two houses along a row and one along a column, other times it will be two houses along a column and one house along a row. I shall call these moves row-knight’s moves and column-knight’s moves respectively. There are four sets where we make a row-knight’s move and four where we make a column-knight’s move. The sets in which we make the same type of knight’s move are generated by the same rule, and therefore represent parallel lines.

It remains to find the rule for locating the other two houses that go with any two houses when we are dealing with using this kind of rule.,

The set can be constructed in three steps, starting with the corner house. The first step is to take one of the knight’s moves. This will determine the second house.

Then we do a one step bishop’s move but so that we end up on the edge of the village but without going into a row or column already “occupied” by the first two houses.

The third step is to move on now with a knight’s move, the same kind as the one with which we started. In this way we determine the fourth house in the set.

Let us give names to our houses, relative to the set within which we are working. We could say:

(a) corner house,

(b) first knight’s house,

(c) bishop’s house,

(d) second knight’s house.

There are six ways of giving two out of the four houses of a set. For each of these ways there must be a method for finding the other two houses of the set. Let me describe these ways:

(a) and (b) Make the bishop’s move followed by the second knight’s move, making sure that the second knight’s move is the same type as the first.

(a) and (c) This means a corner house and a house on a non-adjacent edge 2 steps by 3 steps from the corner house..

Make the knight’s move from the corner which will get you nearest to the other given house. Then from the bishop’s house make the knight’s move to get the fourth house.

(a) and (d) This means a corner house and a house on a non-adjacent edge 3 steps by 1 step away from the corner house. Just make a knight’s move (of the same kind) from each of the given houses to reach the other two houses.

(b) and (c) This means you have two houses a one step bishop’s move away from each other, one of them being on an edge.

Just make a knight’s move from each of the given houses as before.

(b) and (d) This means that one house is on an edge and the other “inside” and they are a knight’s move away from each other. Note the type of knight’s move. From the inside house make a knight’s move of the opposite kind to the kind separating the given houses. This will take you to a corner house. Do the same from the edge house, this will take you to another edge house.

(c) and (d) This means you have two edge houses separated by a knight’s move. Do a bishop’s move from one of these to an inside house, from which do a knight’s move to a corner. These will give you the other two houses of the set.

Now, you can choose any two houses and you should be able to find the other two houses that belong to the set to which the two chosen houses belong.

We have 16 houses in the village, but we also have 5 directions or “rules”. Let me remind you of the five rules, given in terms of where they are in the village:

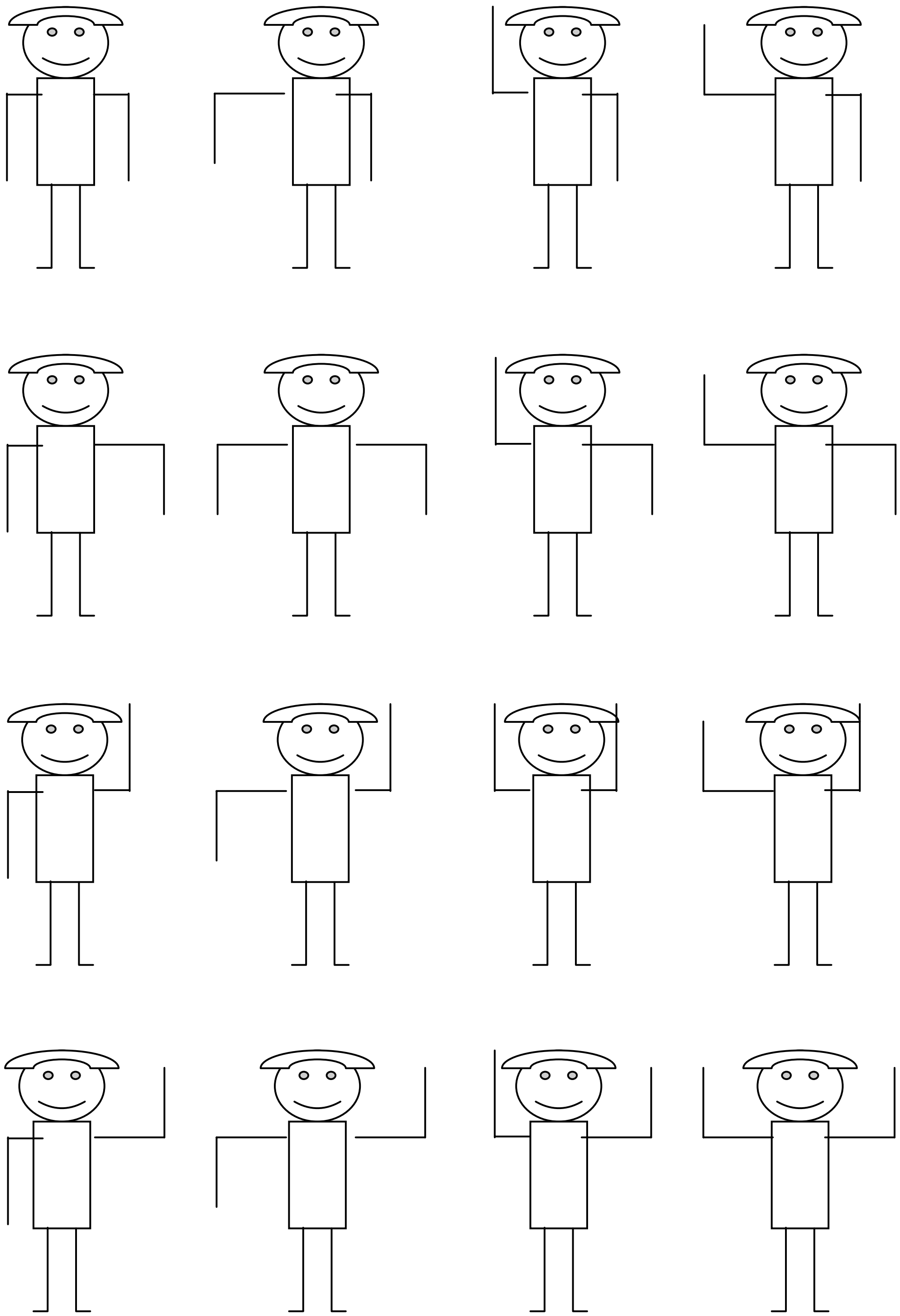
(i) They all belong to the same row
(ii) They all belong to the same column
(iii) They all belong to the same diagonal or “diagonal rectangle”
(iv) They all belong to the same row-knight quadrilateral,
(v) They all belong to the same column-knight quadrilateral.

Given either two houses, or a house and a rule, we can find the missing houses or the missing rule. The five rules can be designated as “houses at infinity”, all of them being in the “set at infinity”. So in this way there are five houses in each set, one of them being “at infinity” So there are 21 sets and 21 houses in all, counting the ones “at infinity”.

Any two sets now have a house in common, but of course the common house might be “at infinity”, if both sets are constructed by the same rule. Also any two houses determine a set, even if one or both are “at infinity”.

Instead of looking at the village, we could look at a set of people holding their arms in different positions Here is a possible way of doing it:



Each arm can be either up or down and either straight or bent.

Here are the simple rules for putting them into sets of four:

(a) The left arm is always in the same position (in a row)
(b) The right arm is always in the same position (in a column)
(c) Arms at same level, either both bent or both straight (diagonal)
(d) Arms at same level, one bent, one straight (diagonal rectangle)
(e) Arms at different levels, both bent or both straight (diagonal rectangle)
(f) Arms at different levels, both bent or both straight (diagonal)

(a) and (b) give us four sets each, the others give us one each. So we have twelve sets already defined.

(g ) Starting from a person in a corner we obtain the other three persons as follows:
  (i) With respect to the corner person, change levels on the left and change bentness on the right. (first knight)
  (ii) With respect to the corner person change levels and bentness on the left and levels on the right. (bishop)
  (iii) With respect to the person in the corner change bentness on the left, and levels and bentness on the right. (second knight)

(h) Do on the left what is done on the right in (g) and do on the right what is what is done on the left in (g).

Since there are four corners, (g) and (h) will give the remaining eight sets.

You might have wondered how we could "add" the arm positions, and what might correspond to redding and greening. So here are some suggestions (based on the rules for adding odd and even numbers):

down + down = down, down + up = up, up + up = down

straight + straight = straight, straight + bent = bent, bent + bent = straight

and we have the two additional operations:

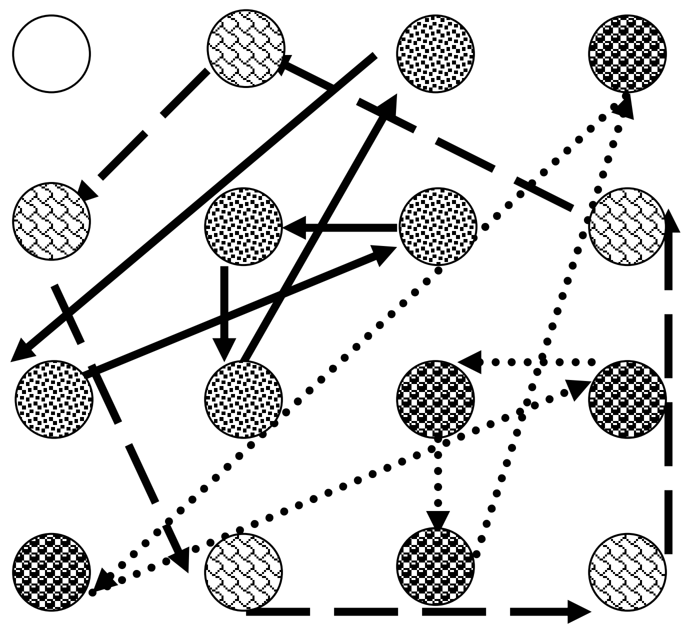
Upping down straight is down straight. Upping down-bent is up bent, upping up bent is up straight and upping up straight is down bent (as we cannot go any "upper"!)

Downing down straight is down straight. Downing up straight is up bent. Downing up bent is down bent. Downing down bent is up straight (as we are not allowed to go any "downer"!).

For example if we wish to split our people into three sets of five (together with the person who has both arms down straight!), we could apply the following rule:

(a) present left position becomes the next right position.
(b) present right arm plus upped left arm becomes the next left position.



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Here is a “map” of our people. It is worth noting that each set of five has a symmetry about the diagonal through the top left corner. You might just like to check my work in determining the “next person” as given by the suggested rule.
Then invent your own rules and see what sets you get. Try to get fifteen people into a sequence by one of your rules.
Also try to work out how the problems with the four colours are linked to the problems with the four arm positions.

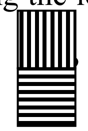
You might want to see how the chess moves are “explained” in terms of changing arm positions. Here are some suggestions:

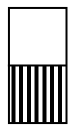
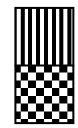
(a) The castle (rook): No change either on the left or no change on the right.
(b) The bishop (one step): In any of the four corner quarters change bentness on both sides. In any of four quarters with twoe edge people, change bentness on one side and level and bentness of the other side. In the square in the middle change bentness and level on both sides.
(c) The knight horizontal: From a corner or for “going round” a corner, change levels on left and bentness on right. From edge to inside change levels on left, change levels and bentness on right.
(d) The knight vertical: From a corner or for “going round” a corner change bentness on left and levels on right. From edge to inside change levels and bentness on left and change levels on right.

We can set ourselves another kind of problem, either with the coloured cubes or with the people in different arm positions. Let us consider two “houses”, using our four colours.
For example take

Can we, either by simply “adding” the houses, or by “adding” either their redded or their greened versions, obtain any other house in the village? “Adding” two houses is done by adding the upper section of one to the upper section of the other, and adding the lower section of one to the lower section of the other.

For example would it be possible to obtain the house 

Redding the first house we obtain  and greening the second house we obtain 

and if we add these two houses, we obtain the house we wanted.

There are two problems you can think about:

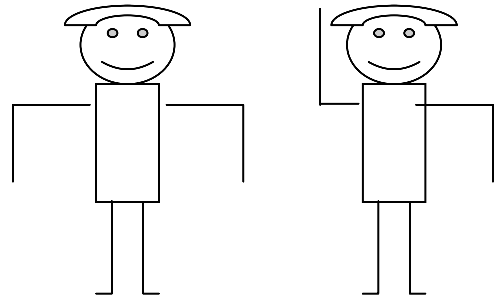
(i) Can you obtain all the houses in the village?
(ii) If so, how can you tell when from two houses you can obtain all the others?

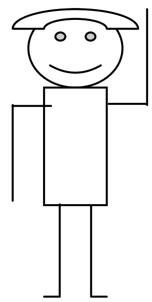
To help you with the first problem, you will have to allow a “yellowing rule”, which turns every house into the all-yellow house.

To help you with the second problem, think of what would happen if one of the given houses were to be a redded or a greened version of the other house?

You might also try to formulate the problem with the persons. How can you tell whether out of two given persons you can obtain any other person by adding them either as they are, or by adding upped or downed versions of them? Of course you ”add” a person to another by adding the left arm of one to the left arm of the other, and the right arm of one to the right arm of the other.

For example, what about starting with these two: 

and trying to make the person  ?

Adding the first to the second one upped will solve the problem. Check it out.

Let me denote "reducing an arm to lower straight" by n, keeping the arm in the same position by s, upping by u and downing by d. We could write the solution of our last problem more concisely as (s, u). If I am not mistaken, we can obtain the other positions as shown in the table below:

| | | | |
|---|---|---|---|
| **(n, n)** | **(d, d)** | **(u, u)** | **(s, s)** |
| **(u, d)** | **(s, n)** | **(n, s)** | **(d, u)** |
| **(s, u)** | **(u, s)** | **(d, n)** | **(n, d)** |
| **(d, s)** | **(n, u)** | **(s, d)** | **(u, n)** |

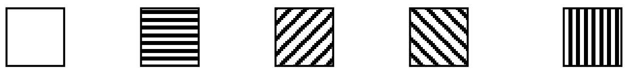
the same arrangement being observed as the one we first had in our four by four table of people. Is there some interest in the above arrangement? You could now choose two other people and see whether you can get all the people from your chosen two and if so, what operations you must perform to do so. You can also do the same thing with the houses.

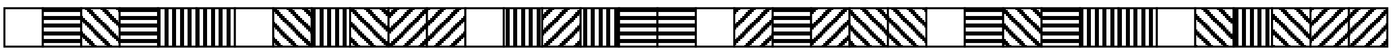
  

# Part V. Five-colour problems with two strips.

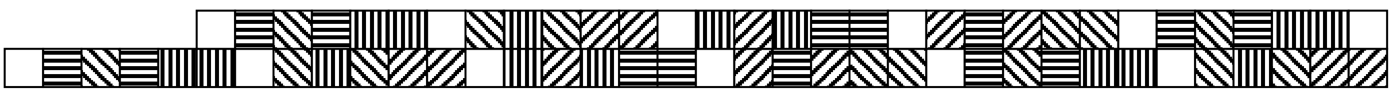
## 1. The strips and the village.

Here are some strips you can use for playing with five colours. You can choose your own colours; I shall just represent them by means of the following patterns:

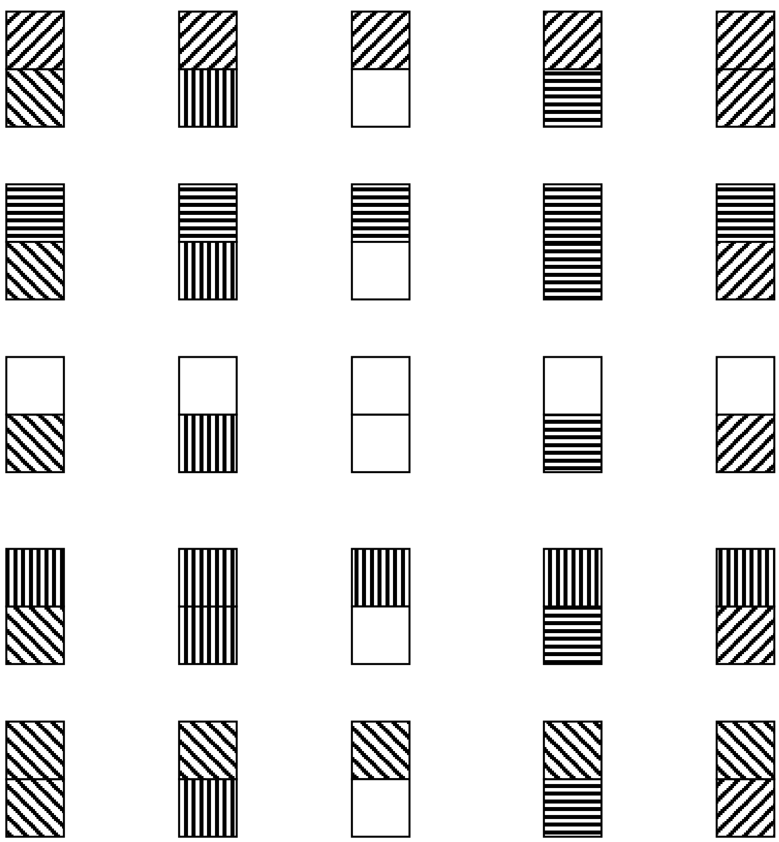




We can use two such strips for sliding one along the other for making sequences. For example we could do a five-slide like this:



We could construct our village as shown below:



# 2. Five-colour operations.

How are we going to “add” and “swap” these patterns?

Of course the sky is the limit to all the possible ways of making up these rules, but let me suggest some that might come in useful in describing the rules which generate sequences.

Imagine a landscape in a remote wild place, and map out a circular tour of some of the interesting features of the place.

There is a desert, by which there is a lake. On the other side of the lake some mountains rise to quite a height. Starting at the foothills there is a big forest that reaches right back to the desert So we can plan our tour in the following way:

(a) We make our way through the desert and arrive at the lake.
(b) We sail across the lake in a local fishing boat.
(c) We climb up to the top of one of the mountains.
(d) We descend, reaching the forest on the other side of the mountain.
(e) We trudge through the forest, finally reaching the desert again.

If we feel like it, we can go on and continue, doing the round trip several times over.
We can use the following symbolic pictures for the various parts of our tour:


These are my suggested rules for “adding”:

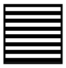
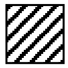
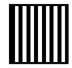
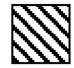
(a) If two patterns to be added contain lake, then the other pattern is advanced by one “notch” along the tour to make the sum.
(b) If two patterns to be added contain forest, then the other pattern is retrogressed one notch on the tour to make the sum.
(c) If two patterns to be added contain desert, then the sum is the other pattern.
(d) Up added to down as well as lake added to forest yield the sum desert.
(e) Up added to up is forest (you climb up and up!), down added to down is lake (you go down and down to the bottom!).

It is perhaps of interest to note that adding any two successive houses in the sequence, using the above rules, we obtain as the sum the house after the next after the second of the pair of houses being considered. So given three consecutive houses of the sequence, we can construct all the rest of the 24 house sequence.

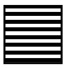
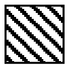
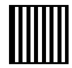
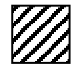
For the “swapping” operations we shall need to make up some different stories.

In one of them we can go climbing trees in the forest. So we could, for example, use the following "story" and call it the climbing operation:

| | | | |
|---|---|---|---|
|  **We are in a clearing within the forest** |  **We go into the forest and start climbing a tree** |  **We get to the top of the tree** |  **We climb down and go to the clearing** |

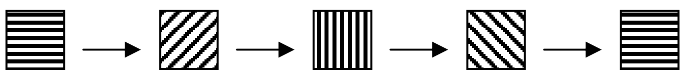
Another "story" could be called diving which could go something like this:

| | | | |
|---|---|---|---|
|  **We are on a lake ready to dive** |  **We dive into the lake and try to reach the bottom** |  **We search for shells at the bottom** |  **We come up to the surface with our shells** |

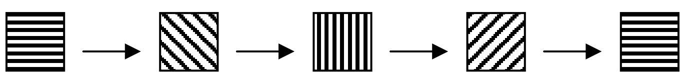
And yet another operation could be called opposites which would exchange the level lines with the vertical lines, as well as the up-lines with the down-lines. You can invent your own story for these, if you like!

So to sum up, we have the following sequences of patterns arising from the above operations:

**Climbing**



**Diving**



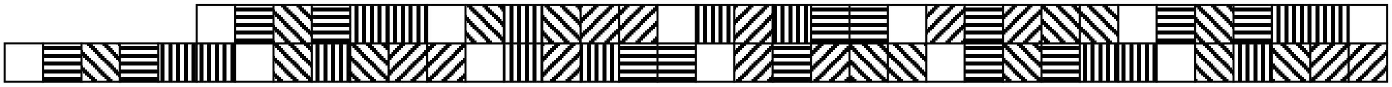
**Opposites**

Applying any of the above operations to the desert will result again in the desert.

# 3. Finding the rules of the sequence.

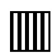
Now let me see if I can "explain" how our sequence develops, using the above operations and the adding.



(a) To obtain the lower section of the next house, just use diving to the upper section of the house you are looking at.
(b) To obtain the upper section of the next house, find the opposite of the lower section of the house you are looking at and add this to the dived version of the upper section of the house you are looking at.

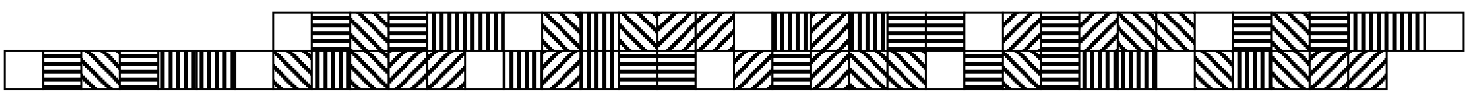
For example try  . If we do a dive on the upper section, we get 

So this will be the lower section of the next house, which it is, as we can check on the sequence.

The opposite of  is  and the dive of  is 

Now adding  to  we get  which is in fact the upper section of the next house.

You could try another sequence, for example



To get the upper section of a house from the previous house, just do a climb on the lower section of the previous house.
To get the lower section of a house from the previous house, just add the upper section to the dived lower section of the previous house.
Make sure you understand how the above two sequences work and then make some of your own by sliding the strips differently and try to find the rules which will generate the sequences you obtain.

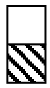
You can also try the trick of generating all the houses out of two given houses, by either adding the given houses or adding their climbed, or dived or oppposited versions.

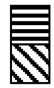
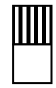
  

# 4. Constructing the village

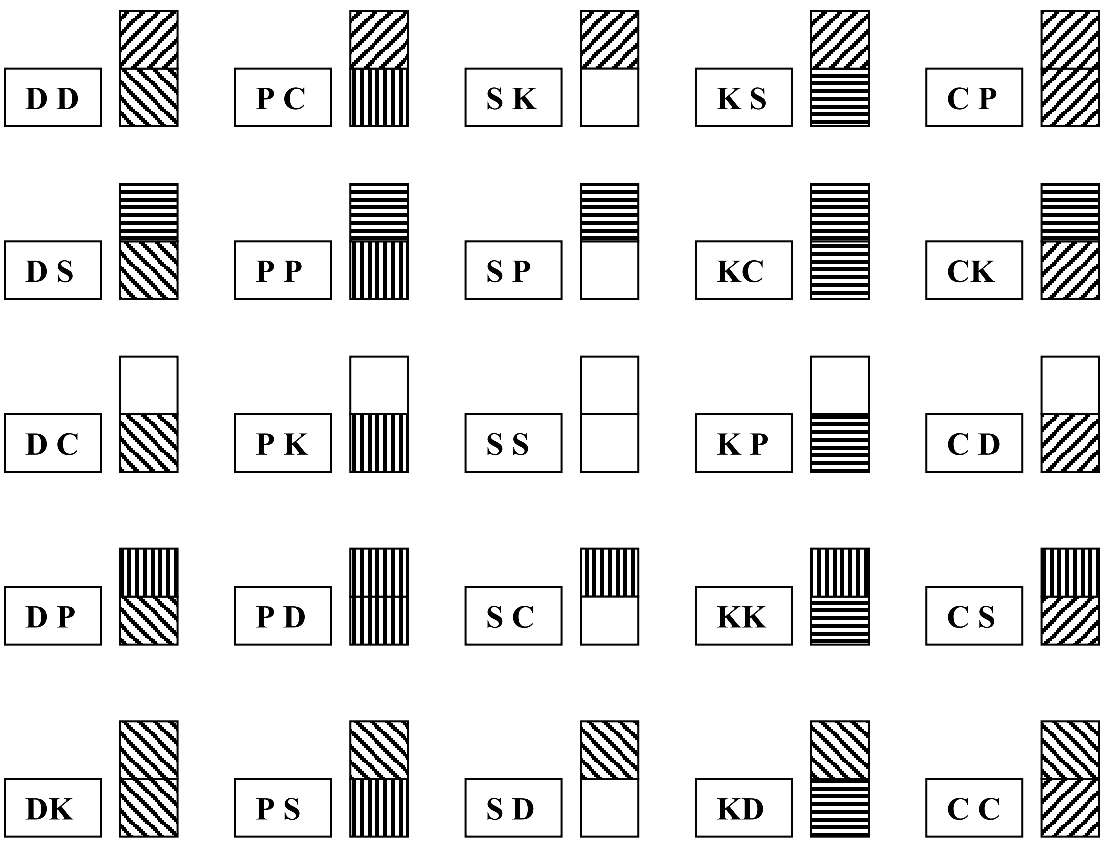
We shall need two more operations: (i) keeping and (ii) sanding or deserting. Keeping will keep any house just as it is, sanding or deserting will turn any house into an all-white house, or expressed more picturesquely, an all-desert house,

Let us pick these two houses: 

How would we "construct"  out of the above two?

If we dive the first one and climb the second one, we shall obtain  and 
and if we add these two, we obtain the desired house.

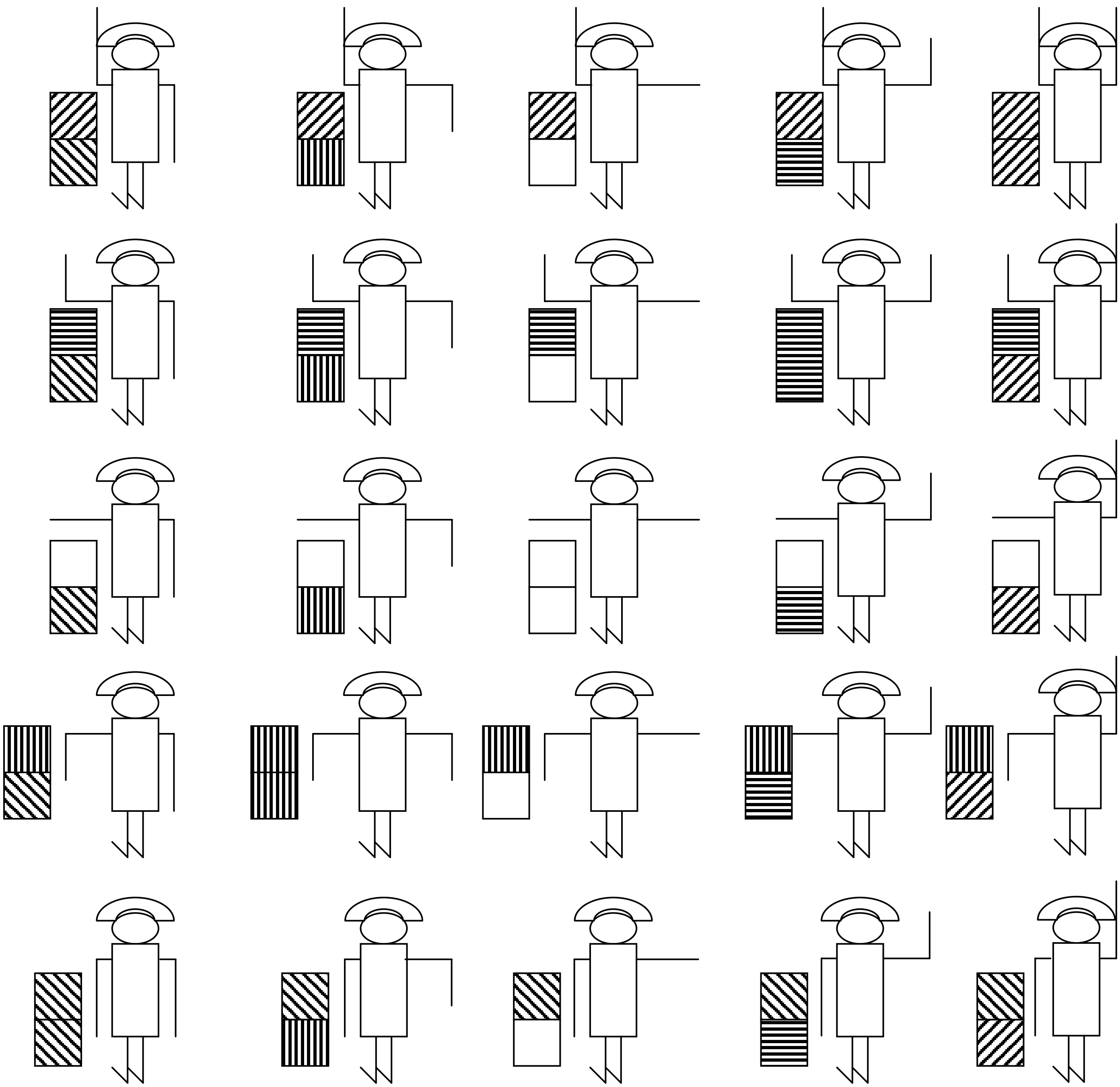
You can operate on each house by deserting it, keeping it, climbing it, diving it or oppositing it after which you must add the resulting houses. So clearly you can do this in 25 ways. If you do this, you will have a table which tells you how to construct each house from the two given houses. If you use S for sanding, K for keeping, C for climbing, D for diving and P for oppositing then the table below will tell you how to construct each house starting with the chosen two. The first letter tells you what you must do to the first chosen house, the second letter referring to the second one. Do not forget that you then have to add the resulting "swapped" houses.



# 5. The village and the people.

Instead of the village of 25 houses we could have a group of 25 people, with arm positions as shown in the diagram below, along with the houses where they "belong":



Arms are either straight or bent and they are placed either below or above the horizontally outstretched position or of course they can be held stretched out horizontally.

We can move an arm upwards or downwards from any position, as long as we decide that going upwards from the highest position means passing on to the lowest position, and that going downwards from the lowest position means passing on to the highest position.
Then the adding rules can be formulated as follows:

(a) If out of two positions one is the horizontal one, then the sum of the two is the other position.
(b) If out of two positions one of them is the upper bent, then the sum is one step upwards from the other position.
(c) If out of two positions one of them is the lower bent, then the sum is one step downwards from the other position.
(d) The sum of two positions in which one is the reflection of the other in the horizontally outstretched arms is always the horizontally outstretched arm.
(e) Straight up added to straight up is bent down; straight down added to straight down is bent up.

For the "swapping operations" we could have the following:

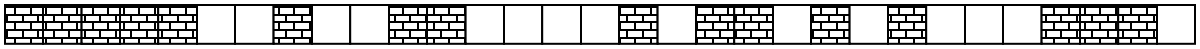
(a) Towards operation: straight becomes bent and stays at the same level, bent becomes straight but goes to the opposite level. (Towards horizontal)
(b) Away operation: straight becomes bent but goes to the opposite level, bent becomes straight and stays at the same level. (Away from horizontal)
(c) Using either the towards or the away on the horizontally outstretched arm, leaves that arm outstretched. (horizontal outstretched beats the others)
(d) Opposite operation: straight remains straight, bent remains bent, but the level is changed. (You obtain the reflection in the horizontally outstretched arm.)

You will see that exactly the same things can be done with the above people operations, as were done with the house operations. You can check this on the diagram on which the houses and the people are placed side by side.

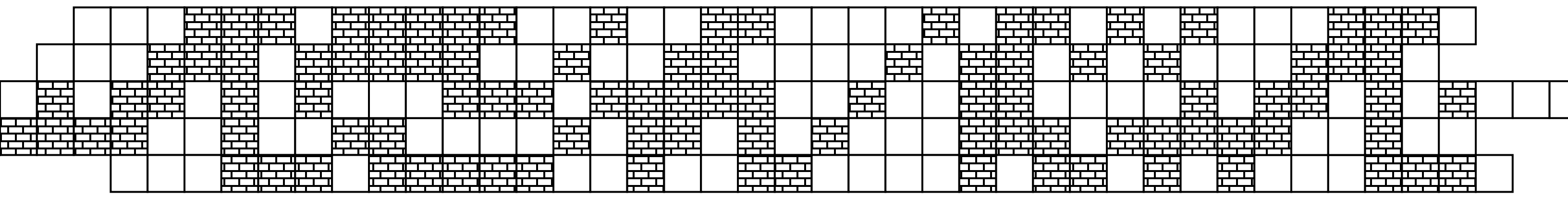
  

# Part VI. Some extra goodies
# Two colors and five strips.

On working with five strips and two colors, the following will be a suitable partial strip:



Below is a suggested way of putting five such strips together: (ignore the ends that go off the page!)



In any column the second color from the top will give the top color next.
The top color added to the third color down will give the second color down next.
The top color added to the fourth color down will give the third color down next.
Adding the top color to the bottom color will give the fourth color down next.
The top color will give the bottom color next.

These arise as a result of the application of the algorithm when 25 multipliers have to be determined. But "guessing" in this case is probably quicker!

It is perhaps worth noting that in any run of six consecutive columns, the first four will always add up to the sixth one.

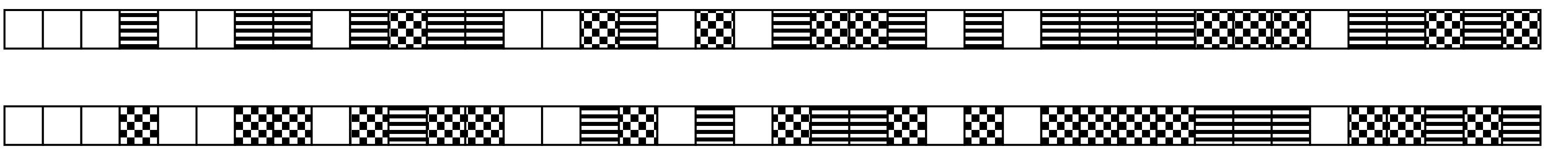
There is also another way of arranging the colors on the strips.

The finding of this arrangement can be an amusing problem for the reader.

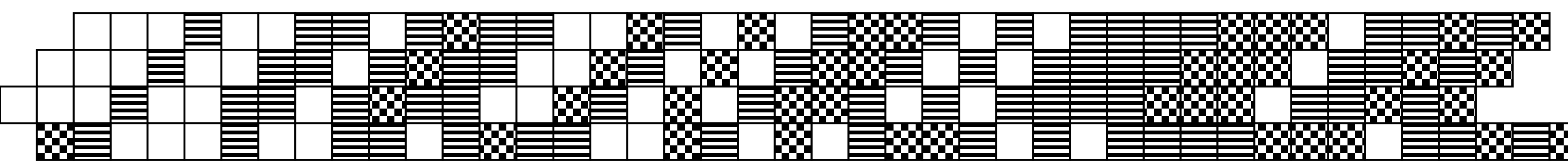
  

# Three colors and four strips.

In this case the strips you will be sliding will be eighty cells long, the second half of forty cells being the inverses of the first forty cells. I am not sure if the strip I am going to give you is the only possible strip for making up four strip sequences with three colors. I will leave that problem for you to solve. If you find another way of arranging the colors on your strip, please send me an e-mail! Below are the two halves that you will have to place end to end to make one whole strip:



Try the following way of sliding four such strips:



We find the four "singles" right at the beginning of the columns. Let (x, y, z, w) denote a column, colors being counted from the top down. So we can see that

$x_1$ = white, $x_2$ = blue, $x_3$ = white, $x_4$ = white

$y_1$ = white, $y_2$ = white, $y_3$ = blue, $y_4$ = white

$z_1$ = blue, $z_2$ = white, $z_3$ = white, $z_4$ = blue

$w_1$ = blue, $w_2$ = white, $w_3$ = white, $w_4$ = white

$$\text{Rule} = \boxed{\begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}}$$

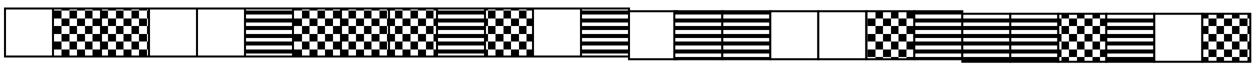
We can use 0 for white, 1 for blue and 2 for red.

Then the color rules will be seen to be equivalent to mod 3 adding and multiplying.

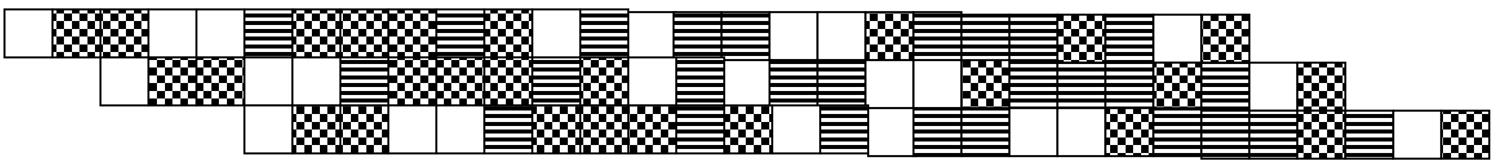
  

# Three strips and three colours.

Here is a strip you could use if you want to use thereof them, using three colours.



Make two of these and stick them together end to end. This will make one of your strips. Then make two more such long strips, so you have three altogether. Then you could slide them along each other in the following way:



This will make a cycle of 26 three-storey “houses”.

The rule for getting the next house is this:
To find the colour of the top floor of the next house add the top and the bottom colours of the house where you are, but swap the resulting colour.
To find the colour of the middle floor of the next house add the middle colour to the swapped lower colour of the house where you are.
To find the colour of the lower floor of the next house, add the swapped upper colour, the middle colour and the swapped lower colour of the house where you are.

Now try to slide your strips differently, but make sure you get no houses with all three floors yellow! Then find the rule for getting the next house from wherever you happen to be.

Possible sequences for three colours and three strips

Writing y for yellow, b for blue and r for red, so far I have found the following two sequences, each of which can of course be reversed

Y R R Y Y B R R R B R Y B Y B B Y Y R B B B R B Y R theme 1

Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B theme 2

If you try to make other cycles out of the above two cycles by taking elements always at the same “distance” from each other in the cycles, provided the length of such a distance does not have a common factor with 26 (namely if it is not even or 13 elements long),
You will obtain either theme 1 or theme 2 or theme 1 reversed or theme 2 reversed.

Taking a theme 2 strip, we can for example slide the second strip two spaces to the right and the third strip four spaces to the right and we shall get a sequence of "houses" with the following succession rules:

Swapped top colour added to swapped bottom colour will give the top colour next.
Swapped top colour added to middle colour added to swapped bottom colour will give the middle colour next.
Swapped top colour added to middle colour will give the bottom colour next.
You can check the above rules on the strips below:
.

```
Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
    Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
        Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
```

You can make a slight change in the above and slide the third strip five spaces to the right instead of four spaces. Then you will get the following set of rules for the next "house":

Add the swapped top, the swapped middle and the swapped bottom to get the next top.
Add the swapped top to the swapped bottom to get the next middle.
Add the middle to the bottom to get the next bottom.

```
Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
    Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
          Y B R B B R Y B B B Y Y R Y R B R R B Y R R R Y Y B
```

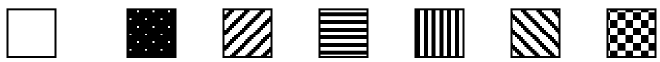
It might be worth noting for example that if you add two consecutive houses, top to top, middle to middle, bottom to bottom, you will find such a "sum" house eight houses on in the sequence.. If you add three consecutive houses, you will find the sum house four houses on in the sequence.

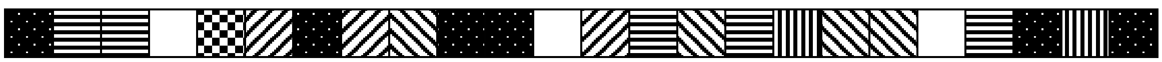
There are some other oddities like that:
For example out of any four consecutive houses the sum of the first, the swapped second and the fourth will always yield the all yellow house.
Try to find some more oddities.

# Two strips and seven colours.

Let the following be our seven colours.





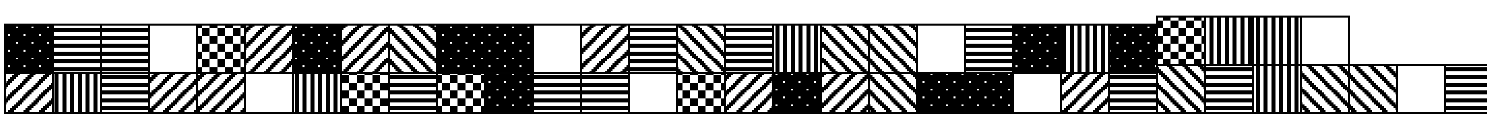
Place these two strips end to end, the second strip being the continuation of the first strip.

You will thus have made your first strip.

Make another one just like your first one

This will be your second strip.

You can no slide them back and forth, keeping one underneath the other, thus making a sequence of "houses" with two floors each every time you decide to stop sliding.
You could, for example, move the top one ten spaces to the left. You will get something that begins like this:

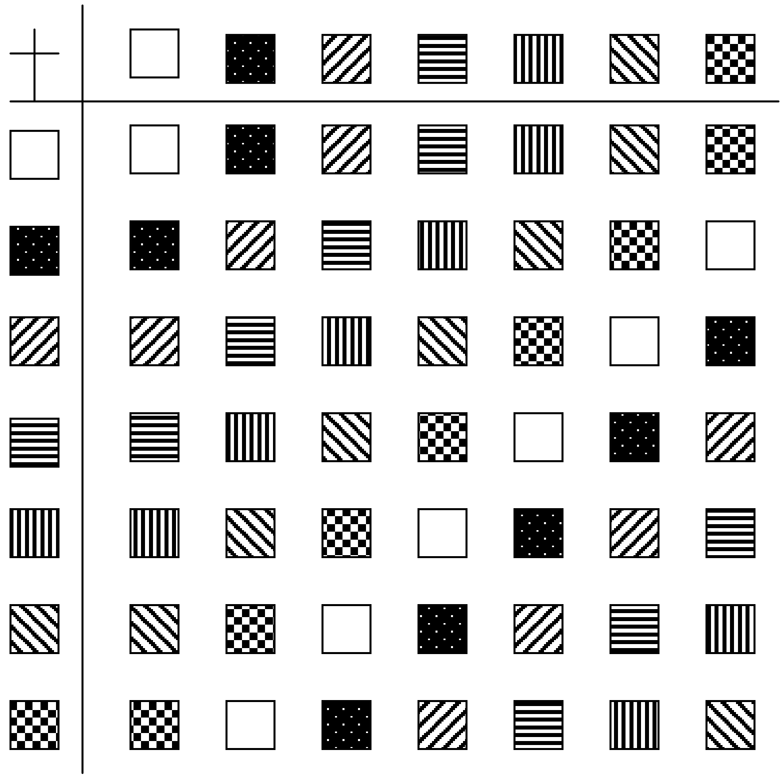


Before we can think about the rule that generates this sequence of houses, we have to agree on the adding rules and on the swapping rules.

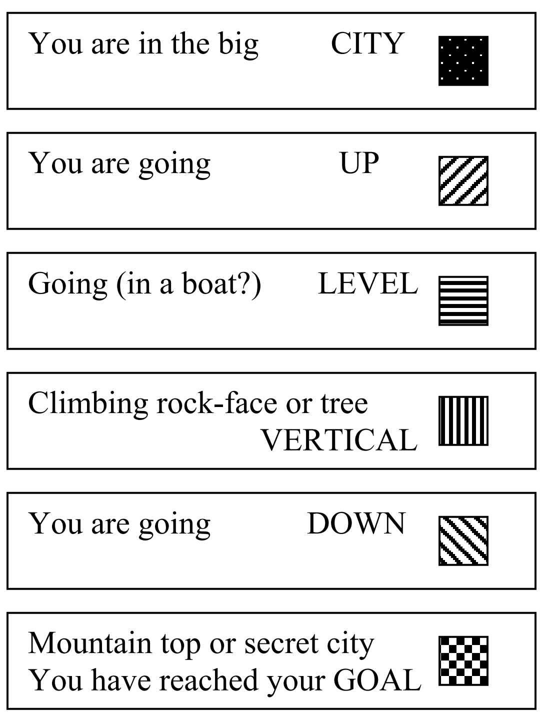
I suggest some rules on the next page.

Adding rules:



You could weave some stories, for example, with the following interpretation of the symbols:



If you can make up a suitable story, then you can remember the sequence of patterns in the first row from left to right. Then the "adding rules" would be as follows:
Adding a white to any pattern leaves that pattern unchanged (as does zero in numbers).
Adding a CITY to a pattern makes that pattern move up one space in the sequence.
Adding an UP to a pattern makes that pattern move up two spaces in the sequence.
Adding a LEVEL to a pattern makes that pattern move up three spaces in the sequence.
Adding a VERTICAL to pattern makes that pattern move back three spaces in the sequence.
Adding a DOWN to a pattern makes that pattern go back two spaces in the sequence.
Adding a GOAL to a pattern makes that pattern go back one space in the sequence.

Then I can suggest the following swapping operations:

WHITING turns every pattern white

BLACKING or CITY-ING leaves every pattern how it is

UPPING does one of these two:
From the CITY you go UP a slope and the climb a VERTICAL rock-face, then fly to the CITY
After boating LEVEL in a lake, you reach your GOAL and run DOWN to the lake to be LEVEL

LEVELLING : From the CITY you take a boat (on LEVEL water!) then go UP some hills till you reach your GOAL (top!), then climb down a VERTICAL rock-face, then run DOWN the hill back to the CITY

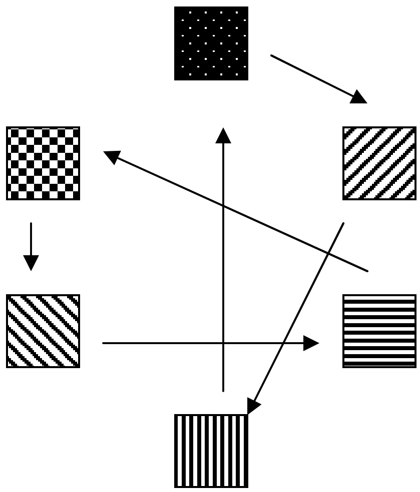
VERTICALLING: One of these two
From the CITY go straight to the VERTICAL rock-face, then UP a bit higher then back to the CITY
From the (LEVEL) lake go DOWN underground to reach GOAL (secret City!), after which quickly return to the (LEVEL) lake.

DOWNING: From the CITY go DOWN in a descending tunnel, then descend in a VERTICAL mine shaft to reach your GOAL (secret city!), then come UP in an ascending tunnel and emerge at the (LEVEL) lake, after which you get back to the CITY.

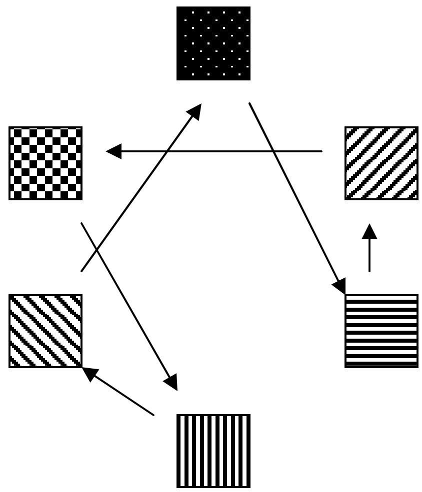
GOALING or OPPOSITE-ING:
The following get INTERCHANGED:
CITY with GOAL
UP with DOWN
LEVEL with VERTICAL

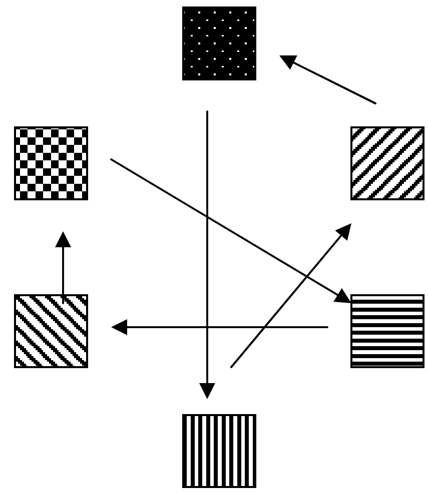
Here is another way of trying to remember our operations.



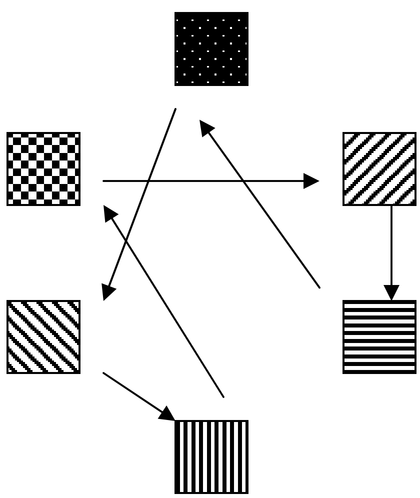
UPPING



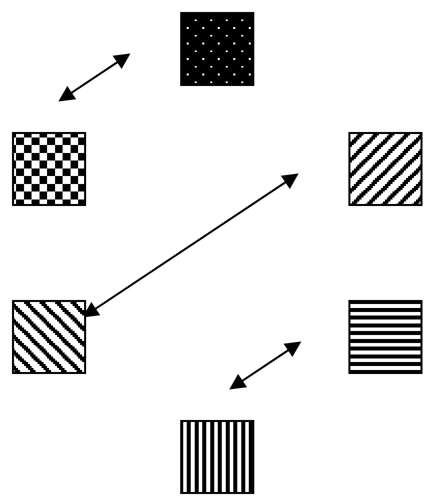
LEVELLING



VERTICALLING



DOWNING



OPPOSITE-ING

Now we can solve the rule problem of our sequence of houses.

To get the pattern of the upper floor of the next house just add the verticalled top to the levelled bottom of the house in which you are.

To get the pattern of the lower floor of the next house just add the upped top to the bottom of the house in which you are.

Now try and slide your strips into a different position, making sure you never have a white on top of a white. Then try to find the rule of the next house in the sequence you have generated. Have fun!

# Some notes on three strips and four or five colours.

**One way to start a good strip that will serve well for using three of them is to start with a yellow, followed by a blue, followed by a red. I then suggest you add these and red the sum.**

Since adding blue to yellow is green, and redding green yields blue, our fourth colour will be blue. Now we have the three colours blue, red and blue to add, which yields red, which when redded yields green, which will be our fifth colour. Going on like this we obtain a strip with sixty-three consecutive colour elements. Split into three parts, the strip will look something like this (using letters now instead of patterns, to save ink!)

```
Y B R B G Y G Y B G G R G G G B R Y B B Y

Y R G R B Y B Y R B B G B B B R G Y R R Y

Y G B G R Y R Y G R R B R R R G B Y G G Y.
```

Now suppose we do some very simple sliding like this

```
    Y B R B G Y G Y B G G R G G G B R Y B B Y Y R G R B Y B
  Y B R B G Y G Y B G G R G G G B R Y B B Y Y R G R B Y B
Y B R B G Y G Y B G G R G G G B R Y B B Y Y R G R B Y B
```

The rule for getting the column next to any desired column is this:

The middle colour in a column always gives the top colour of the next column.
The bottom colour in a column always gives the middle colour of the next column.
If you add all three colours in a column and red the result, you will always get the bottom colour of the next column.

It is also the case that if you add three consecutive columns, top to top, middle to middle and bottom to bottom, and then red the entire resulting column, you will get the column immediately following the three columns you have chosen .

I suggested the above rules at  the end of Chapter XI of my book “I will tell you algebra stories you’ve never heard before”, but not with colours but with trees and star fishes, some thin, some dotted and some thick, to distinguish the “dimensions” from each other.

If we want to use five colours, it is perhaps better to stay with the patterns :

CLEARING = O, LEVEL = L, UP = U, DOWN = D, VERTICAL = V

I suggest the following way of making three five-pattern strips:

Start with Level – Up – Down.
Up the first and add to the second to get the fourth, giving Up + Up = Vertical,
So the fourth pattern is Vertical.
So we have: Level – Up – Down – Vertical so far.
Now Up the second and add to the third, and since V + D = U, so the fifth one is Up.
So we now have: Level – Up – Down – Vertical – Up
So always looking at the last three, Up the first of these and add it to the second, to get the next pattern..
If you have the patience, you will eventually have a strip with 124 patterns on it!
You can then make three of these and slide them quite simply as follows :

```
    L U D V U O O V 0 V D V L O V U V O D D D V V 0 U D U U D L U
  L U D V U O O V 0 V D V L O V U V O D D D V V 0 U D U U D L U
L U D V U O O V 0 V D V L O V U V O D D D V V 0 U D U U D L U
```

The middle pattern always gives the next top one,  the bottom pattern the next middle one and the upped top added to the middle pattern always gives the next bottom one.

Out of any three consecutive columns upping the first column and adding it to the second will always yield the column that comes after the three chosen ones.

I think I might leave you the pleasure of working out the strips for seven patterns unless you are willing to toil away with these suggestions:
Start making a strip putting  white – white – up in a row and always calculate the fourth pattern after every three by adding the upped first pattern to the second. I believe you might get a strip 343 patterns long! Then you might even try the same sliding trick used for the five patterns

You may also ponder over what might happen if you tried six patterns. I have an idea you might have problems!

# APPENDIX 1
# Mathematical notes on the activities with the strips.

## 1. What is a group?

In all the activities with the strips we used either two or three or four or five or seven different colours. Each time, before beginning any other activity, a form of "addition" was suggested, in which the various colours were the addenda. It is worth noting that in each case "adding a colour" satisfied the following conditions:

**(a) Adding a colour to any allowed colour, always resulted in one of the allowed colours (we call this property closure, "new" colours were never produced).**

**(b) Adding a colour A to a colour B always yielded the same colour as adding the colour B to the colour A (we call this property the commutative property).**

**(c) In the case of any three addenda A, B and C, the sum of A and B added to C always yielded the same colour as A added to the sum of B and C (we call this property the associative property).**

**(d) There was always a colour, usually yellow or white, which when added to any other colour, always yielded this other colour (we call this property the existence of a neutral colour or additive neutral).**

**(e) To every colour there was always a corresponding colour (another colour or the same one) such that when added, the sum of the two colours was the neutral colour (we call this property the existence of an inverse colour)**

Any set of elements (colours, shapes, symbols of any kind) in which an "addition " is defined which satisfies conditions (a), (c), (d) and (e) is called a **GROUP**. If condition (b) is also satisfied, the group is called **ABELIAN**.

All the additions suggested in the activities form Abelian groups.

For the three colour activities we can use yellow = 0, blue = 1, red = 2,
For the four colour activities yellow = 0, blue = 1, red = 2, green = 3

the addition tables for the three colours and for the four colours are these:

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

It is easy to see that the three colour addition is addition in mod3 and the four colour addition is just like mod8 multiplication of the first four odd numbers.
The five colour addition follows the same rules as mod5 addition. With seven colours we have mod7 addition.

The two colour addition follows mod2 addition and so comes to the following:

$$\mathbf{0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0}$$

where 0 symbolizes white and 1 symbolizes black.

# 2. What is a field?

In all except the two-strip activities, an operation that was called “swapping” was defined. This operation is usually known in mathematics as multiplication.

If for the set of all the colours less the neutral one (yellow or white) the “swapping operation” is a group operation satisfying conditions (a) to (e), then we have the makings of a field. Before we can be sure that our colours (elements) form a field, there is a property which connects “swapping” and “adding”, which must be true as well. Denoting the swapping or multiplying operation by means of a star, this property is the following:

**(D) For any three colours (elements) A, B and C it must be true that**

$$\mathbf{A * ( B + C ) = (A * B) + (A * C)}$$

In other words the colour obtained on the left must be the same colour as the one obtained on the right of the equal sign above.

This property is called the DISTRIBUTIVE PRPOPERTY, a well known property connecting the arithmetical operations of addition and multiplication.

One more thing: multiplying by or multiplying the additive neutral should always yield the additive neutral! In other words

**yellow * any colour = that colour * yellow = yellow**

So if the addition rules form an Abelian group, and the multiplication rules applied to the colours (elements) not including the additive neutral also form an Abelian group and if the distributive property is true, then our colours (elements) form a FIELD. In the vernacular this merely means that all the rules that are true for example for positive and negative rational numbers are also valid for our prospective field. The elements of a field are also called SCALARS.
All the “swapping rules” suggested in the activities lead to fields.
There is a miserable kind of multiplication even in the two element field, but since one colour has to be the additive neutral, there is just one colour (black) left to use in our “swapping group”.

This merely states that

**black * black = black.**

The multiplication rules ("swapping") in the three colour activity follow the rules of multiplication in mod3, whereas for the five or for the seven colours, they follow the rules of multiplication in mod5 and in mod7 respectively. Since in the case of four colours we do not follow the mod4 rules, but the rules of the so called two by two group or Klein group, the multiplication is defined by cyclic changes. The element 1, corresponding to blue, is the neutral for multiplication, in other words it behaves like a normal 1 in arithmetic. But multiplying by 2 (red) changes the multiplicand round the cycle

$$\mathbf{1 \rightarrow 2 \rightarrow 3 \rightarrow 1}$$

and multiplying by 3 (green) uses the reverse cycle, namely makes the multiplicand go through the cycle:

$$\mathbf{1 \rightarrow 3 \rightarrow 2 \rightarrow 1}$$

The field thus generated is not a prime field, but it is usually referred to as a Galois field, after the famous but tragically short lived mathematician of that name.
When we use five colours (or patterns), we make use of the five element field. In the case of seven patterns, we have recourse to the seven element field.
The five element field used modulo 5 addition for its additive group, and modulo 5 multiplication for its multiplicative group. It will be noticed that

Desert = 0, lake = 1, up = 2, down = 3. forest = 4

So "climbing" or "upping" is multiplying by 2,

then "diving" or "downing" is multiplying by 3,

and "oppositing" or "verticalling" is multiplying by 4.

In the "story line" "laking" or "levelling" is left out, since it does not do anything
As well as "deserting" or "annulling" are not in the story, since instead of playing the disappearing trick, it is perhaps better for things not to appear in the first place!

You can no doubt work out how the "numbers" 0, 1, 2, 3, 4, 5, 6 in the seven element field correspond to the "story elements".

# 3. Vectors.

Given a field, a vector is a set of a certain given number of elements of the field, in a certain given order. In other words a vector is an ordered set of scalars.

The number of dimensions of a vector is the number of scalars that compose it.

The set of all possible vectors of a given dimension that can be constructed out of the field in question is called a **VECTOR SPACE** of that same number of dimensions. The way we obtain other vectors from given vectors is by adding vectors or by multiplying a vector by a scalar.

In our activities we have used fields with two elements, three elements, four elements five elements and seven elements. They were represented by two, three, four and five and seven different colours or patterns respectively. The vectors were constructed by using the strips. The number of strips used gave the number of dimensions of the vector space. Each vector was a column of colours, read from top to bottom. As we read the colours in a column from top to bottom, we read the components of the vector represented by that column. An n-dimensional vector would have n components, using n strips, which could be symbolized in this way:

$$(x_1, x_2, x_3, \ldots x_n)$$

the first one representing the top colour, the second one the next colour down, and the last one representing the bottom colour.

To add two vectors we just add the corresponding components. For example

$$(x_1, x_2, x_3, \ldots x_n) + (y_1, y_2, y_3, \ldots y_n) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots x_n + y_n)$$

We can also multiply a vector by a scalar. For example if “a” is a scalar, we have

$$a * (x_1, x_2, x_3, \ldots x_n) = (a * x_1, a * x_2, a * x_3, \ldots a * x_n)$$

where each component is multiplied by the scalar (or “swapped by the scalar”, in the language of our activities!) .

A set of vectors are said to be (linearly) dependent if at least one of them can be expressed in terms of some or all the rest of the vectors as a linear combination. For example if U, V and W are vectors and W is the one that can be expressed in terms of U and V by means of scalars a and b of which at least one is a non-neutral scalar in other words if

$$W = a * U + b * V$$

then U, V and W are (linearly) dependent, because one of them, here W, has been expressed linearly in terms of U and V.

If no such relationship is possible with non-zero scalars, then the vectors are said to be (linearly) independent.

# 4. Matrices.

If we wish to "transform" a vector into another vector, we can use what is known as a **LINEAR TRANSFORMATION**.

This means that we decide how each component of the transformed vector will arise as a linear combination of the components of the vector we wish to transform. Let us take an example from the work with three strips, namely from a three dimensional vector space. Here are the suggested rules:

**(a) The middle colour of a house will give the upper colour of the next house.**

**(b) Adding the middle colour to the "swapped" lower colour of a house will give the middle colour of the next house.**

**(c) Adding the colours on all the floors will give the lower colour of the next house**

The "**house**" we want to transform is the one we are looking at. Let the colours be denoted by x, y and z, counted from top to bottom. Let the colours of the transformed house (vector) be X, Y and Z. Let us remember that yellow is 0, blue is 1 and red is 2. Leaving a colour as it is can be symbolized by a multiplication by 1, which can be called "**blueing**" while the "swapping" could be called "**redding**", since "swapping" a colour is a multiplication by 2, since it "swaps" red and blue, namely 2 and 1, remembering that 2 * 2 = 1 in this field. We should also remember that multiplying by 0 will yield 0. This can be, in the game, the "**yellowing**" operation. So the transformation rules can be expressed algebraically as follows:

$$\mathbf{X = 0 * x + 1 * y + 0 * z}$$

$$\mathbf{Y = 0 * x + 1 * y + 2 * z}$$

$$\mathbf{Z = 1 * x + 1 * y + 1 * z}$$

| | | |
|---|---|---|
| **0** | **1** | **0** |
| **0** | **1** | **2** |
| **1** | **1** | **1** |

which can be abbreviated to the above three by three table known in mathematics as a **matrix**.

# 5. Subspaces

A set of linearly independent vectors from which all vectors of the vector space can be constructed as linear combinations of the vectors of the set, is called a **BASIS** of the vector space. There will always be more than one possible basis for any given vector space.

If only a part of the space can be reached by making linear combinations of the vectors in the set, then the set of the vectors reached is called a subspace.

For example let us think of the three dimensional vector space based on the two element field. The vectors in this space are:

**(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)**

If we try the three vectors **U = (1, 1, 1), V = (1, 1, 0), W = (0, 0, 1)** as a basis, we shall be unlucky as the sum of any two of these is always the third one and the sum of all three is (0, 0, 0), so there is no way we can construct the remaining four vectors starting with the given ones. So the set **{U, V, W, }** consists of three **LINEARLY DEPENDENT** vectors. But if we choose

**U = (1, 1, 1), V = (1, 1, 0), Q = (1, 0, 1),** we can derive all the vectors in the space in the following way:

**U + V = (0, 0, 1), V + Q = (0, 1, 1), U + Q = (0, 1, 0)** and

**U + V + Q = (1, 0, 0)**

To get (0, 0, 0) we merely add any one of the three to itself.

Here is a way of getting the seven subspaces and the seven non-zero elements to correspond to each other:

**[001, 010, 011] → 110**

**[010, 100, 110] → 010**

**[100, 011, 111] → 111**

**[011, 110, 101] → 001**

**[110, 111, 001] → 011**

**[111, 101, 010] → 100**

**[101, 001, 100] → 101**

You can check that for any three subsets with a vector in common, the three corresponding vectors in the right hand column form one of the subsets.

The subsets could be thought of as lines and the vectors as points in a seven point geometry. The points corresponding to three concurrent lines are always collinear. Points and lines are in a dual situation.

You will now recognize some of the sets of houses in a "picnic party" as forming a subspace of the vector space in which you are working.

You should also recognize the rules for getting from vector to vector (from house to house) as matrices. Each of these matrices, when applied to any non-zero vector seven times in succession, will bring you back to your starting point, having "visited" all the vectors except (0, 0, 0). These are the "walks through the seven house village", each one consisting of seven steps. We say that the matrix is of order seven.

There are seven non-zero vectors and also seven subspaces. As we have just seen, they can be put into one to one correspondence with each other, in such a way that three vectors belonging to a subspace correspond to three subspaces with a vector in common. This was pointed out in the section on "helpers"

When we pass on to four dimensions (with the four strip situations), there will be subspaces of four vectors as well as subspaces of eight vectors.

In the activities I usually ignore the vector (0, 0, 0, 0), so it looks (wrongly) as though the subspaces had three and seven vectors in them respectively. These are of course the vectors other than (0, 0, 0, 0) or the zero vector, that form part of the subspace in question. Here there are fifteen non-zero vectors altogether, but also fifteen seven-element subspaces, so these can be placed in one to one correspondence with each other. This can be done in such a way that for any three seven-element subspaces with a three-element subspace in common, the corresponding associated vectors themselves form a three-element subspace. The realization of such relationships is a good introduction to the idea of **DUALITY**.

A basis in this four dimensional case would have to consist of four linearly independent vectors. This is shown in the part where the "village" is constructed from four houses, naturally so chosen that they form a set of linearly independent vectors! The vectors in this case were

$$\mathbf{(1, 0, 0, 0),\ (1, 0, 1, 0),\ (1, 1, 1, 1),\ (1, 1, 1, 0)}$$

and it is easily verified that all the vectors in our space can be constructed by adding some or all of the above four together.

A matrix which takes through all the vectors of the space except the zero vector is

$$\boxed{\begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{matrix}}$$

**0011, 0101, 1001, 0001, 0110, 1100, 1000, 0111,1010, 0100, 1111, 1101, 1110, 1011, 0010**

being the sequence generated, as you can check from the corresponding part of the described activity by sliding the four strips into the appropriate positions.

Let me go through the subspaces in a case in which we have used four colours, but only two strips. So we are dealing with a two dimensional vector space, the basis field being the Galois field of four elements, which I shall call 0, 1, 2 and 3. Since we have a two dimensional vector space, the only subspaces can be one dimensional. These are obtained by taking any one vector and finding all the vectors that are multiples of this vector by a scalar. If x denotes the "upper colour" (first component) and y the "lower colour" (second component) of a house in the "village" (vector space), then the subspaces each have a property which is a linear combination of x and y. These are the combinations:

**(a) $x = 0$, (b) $y = 0$, (c) $x + y = 0$, (d) $2 * x + y = 0$, (e) $3 * x + y = 0$**

Each one of these includes the zero vector (0, 0) where $x = 0$ and $y = 0$.

If we wish to divide the vector space into four sets of four vectors, all we have to do is to write 1, 2 and 3 respectively on the right hand sides of the above equations. These will give you the interesting patterns found in our work with the "village".

# Appendix 2.

## Rules for making the strips.

### 1.Using two strips.

#### 1a. Using two colours: yellow and blue.

Addition rules: Y + Y = Y, Y + B = B, B + Y = B, B + B = Y

Start with any two cubes, except Y Y.
In any run of three cubes, the sum of the first and the second must be equal to the third.

FOR EXAMPLE WE COULD HAVE

```
B R R B R R B
  B R R B R R B

top next     =  top  +  bottom

bottom next  =  top
```

#### 1b. Using three colours: yellow, blue and red.

Addition rules: Y + Y = Y, Y + B = B, Y + R = R

B + Y = B, B + B = R, B + R = Y

R + Y = R, R + B = Y, R + R = B

Swapping rules: S Y = Y, S B = R, S R = B

Start with any two cubes, except Y Y .
In any run of three cubes the sum of the first and second colours must be the colour of the third cube.
FOR EXAMPLE WE COULD HAVE:

```
B R Y R R B Y B B R Y R R B
    B R Y R R B Y B B R Y R R B

top next      =  swap top  +  swap bottom

bottom next   =  top       +  swap bottom
```

# 1c. Using four colours: yellow, blue, red and green.

Addition rules: Y acts as the "zero". Any two cubes of the same colour will have Y as their sum. Any two non-yellow cubes add up to the non-yellow cube of the third colour.

Swap rules: "Redding" Red Y = Y, Red B = R, Red R = G, Red G = B

"Greening": Gr Y = Y, Gr B = G, Gr G = R, Gr R = B

Start with any two cubes except Y Y,

In any run of three cubes, the redded colour of the first cube and the colour of the second cube must add up to the colour of the third cube

FOR EXAMPLE:

```
B R Y G G R G Y B B G B Y R R
    B R Y G G R G Y B B G B Y R R

top next     =     Green top   +   Green bottom

bottom next  =     top         +   Red bottom
```

# 1d. Using five patterns: black, level, up, down, vertical (B, L, U, D, V).

Addition rules: (Adding cycle is B → L → U → D → V → B)

(i) Adding B to a pattern leaves that pattern unaltered.
(ii) Adding L to a pattern moves that pattern up one step in the cycle.
(iii) Adding U to a pattern moves that pattern up two steps.
(iv) Adding D to a pattern moves that pattern back two steps.
(v) Adding V to a pattern moves that pattern back one step

Swapping rules:

UPPING (or CLIMBING): B → B, L → U → V → D → L

DOWNING (or DIVING): B → B, L → D → V → U → L

OPPOSITE-ING (VERTICALLING): B → B, L → V → L, U → D → U

To make a strip just start with any two patterns except two black ones.
In any run of three patterns the climbed first pattern added to the dived second patterns should be equal to the third pattern.

FOR EXAMPLE:

```
B L D L V V B D V D U U B V U V L L B U L U D D
      B L D L V V B D V D U U B V U V L L B U L U D
```

To obtain the column next to any given column we have the rules:

Add the verticalled upper pattern to the dived lower pattern to get the next upper pattern.
Add the upper pattern to the verticalled lower pattern to get the next lower pattern.

# 2. Using three strips.

## 2a. Using two colours: yellow and blue.

Addition rules: Same as with two strips.

Start with any three coloured cubes, except Y Y Y.

In any run of four cubes the colour of the first cube and the colour of the second cube must add up to the colour of the fourth cube.

FOR EXAMPLE:

```
B B Y Y B Y B B B Y Y B Y B
    B B Y Y B Y B B B Y Y B Y B
        B B Y Y B Y B B B Y Y B Y B

top next     =   top  +  bottom

middle next  =   top  +  middle  +  bottom

bottom next  =   top  +  middle
```

## 2b. Using three colours: yellow, blue and red.

Addition and swapping rules: same as with two strips.

Start with any three coloured cubes, except Y Y Y.

In any run of four cubes

the swapped first colour plus the second colour plus the swapped third colour must be equal to the fourth colour.

FOR EXAMPLE:

```
        Y Y B R R R B R Y B Y B B Y Y R B B B R B Y
Y Y B R R R B R Y B Y B B Y Y R B B B R B Y R Y R R
      Y Y B R R R B R Y B Y B B Y Y R B B B R B Y R
```

top next = bottom

middle next = middle + Swap bottom

bottom next = top + Swap middle + bottom

## 2c. Using four colours: yellow, blue, red and green.

Addition and swapping rules: same as with two strips.

Start with any three coloured cubes, except Y Y Y.

In any run of four cubes redding the sum of all the first three colours should be equal to the fourth colour. You are going to get a very long strip, it will contain 63 cubes before it begins to repeat!

Here are the first 21 you might obtain:

Y B R B G Y G Y B G G R G G G B R Y B B Y

AND IT WILL CONTINUE BY “GREENING” ALL THE ABOVE, NAMELY AS

Y G B G R Y R Y G R R B R R R G B Y G G Y

AND “GREENING” THE ABOVE IT WILL FINISH THE JOB AS

Y R G R B Y B Y R B B G B B B R G Y R R Y

One sliding example could be:

```
Y B R B G Y G Y B G G R G G G B R Y B B Y
  Y B R B G Y G Y B G G R G G G B R Y B B Y
      Y B R B G Y G Y B G G R G G G B R Y B B Y
```

top next = Gr top + Red bottom

middle next = top

bottom next = Gr top + middle + bottom

## 2d. Using five patterns: Black, Level, Up, Down, Vertical (B, L, U, D, V).

The adding rules and swapping rules are the same as for two strips.

To make a strip, start with any three patterns except B B B
Make sure that in any run of four patterns the climbed (upped) first pattern, added to the second one, should be equal to the fourth one

FOR EXAMPLE WE COULD HAVE A STRIP (IN FOUR SECTIONS AS THE STRIP IS TOO LONG TO FIT INTO A LINE!):

```
B B L B L U L V B L D L B U U U L L B D U D D U V D D L V U L
B B U B U V U D B U L U B V V V U U B L V L L V D L L U D V U
B B V B V D V L B V U V B D D D V V B U D U U D L U U D L D V
B B D B D L D U B D V D B L L L D D B V L V V L U V V D U L D
```

```
We could try the following slide (not putting the whole strips
in for lack of space)
```

```
B B L B L U L V B L D L B U U U L L B D U D D U V D D L V U L
  B B L B L U L V B L D L B U U U L L B D U D D U V D D L V U L
      B B L B L U L V B L D L B U U U L L B D U D D U V D D L V U
```

The rules for getting the next column appear to be these:

Add the upped top to the sum of the  middle and of the bottom, you will get the top of the next column.

The top will be the middle of the next column.

The sum of the top and of the downed bottom will be the bottom of the next column.

# 3. Using four strips.

## 3a. Using two colours: yellow and blue.

Addition rules: same as with two strips.

Start with any four colours, except Y Y Y Y.

In any run of five cubes, the sum of the first two colours must be equal to the fifth colour.

HERE IS AN EXAMPLE

```
R R R R Y Y Y R Y Y R R Y R Y
    R R R R Y Y Y R Y Y R R Y R Y
      R R R R Y Y Y R Y Y R R Y R Y
          R R R R Y Y Y R Y Y R R Y R Y
```

top next = upper middle + lower middle

upper middle next = top + lower middle + bottom

lower middle next = upper middle

bottom next = top + lower middle

## 3b. Using three colours: yellow, blue and red.

Addition and swapping rules: same as with two strips.

Start with any four colours, except Y Y Y Y.

In any run of five cubes, the first two colours must add up to the fifth colour.

This strip will not start repeating until you have placed eighty cubes in your strip. It could go for example like this:
YYYBYYBBYBRBBYYRBYRYBRRBYBYBBBBRRRYBBRBRYYYRYYRRYRB

AND YOU COULD PLACE FOUR STRIPS LIKE THIS:

```
YYYBYYBBYBRBBYYRBYRYBRRBYBYBBBBRRRYBBRBRYYYRYYRRYRB
 YYYBYYBBYBRBBYYRBYRYBRRBYBYBBBBRRRYBBRBRYYYRYYRRYRB
  YYYBYYBBYBRBBYYRBYRYBRRBYBYBBBBRRRYBBRBRYYYRYYRRYRB
   YYYBYYBBYBRBBYYRBYRYBRRBYBYBBBBRRRYBBRBRYYYRYYRRYRB
```

top next = lower middle + bottom

upper middle next = top

lower middle next = upper middle

bottom next = lower middle.

You can set yourself more interesting problems by sliding your strips a lot more than I have done in this example. But make sure there are no columns consisting entirely of yellow cubes!

## 3c. Using four colours.

I have not yet worked out the rules for this case.
You can find out for yourself as part of your home assignment.

# 4. Using five strips

## 4a. Using two colours: yellow and blue.

Addition rules: same as with two strips.

To make a strip you can use for five-strip problems, start with any sequence of five cubes except Y Y Y Y Y. Then make sure that in any run of six cubes, the colour of the sixth cube is the colour you get by adding the first four colours of the run. You can start with any five (except five yellow ones), because your strip will contain all possible sequences of five yellow and/or blue cubes. So for example, to tempt providence, let us start with
Y Y Y Y B and see what happens.

```
Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B B
```

We can try the following slides (ignoring the parts off the margin!):

```
Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B B
  Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B B
    Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B B
        Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B B
      Y Y Y Y B Y B B Y B Y B Y Y Y B B B Y B B B B B Y Y B Y Y B
```

Calling the rows a, b, c, d and e, from top to bottom, we have the following rules for calculating the column next to any given column:

a next = b + c + d + e

b next = a

c next = b

d next = e

e next = c

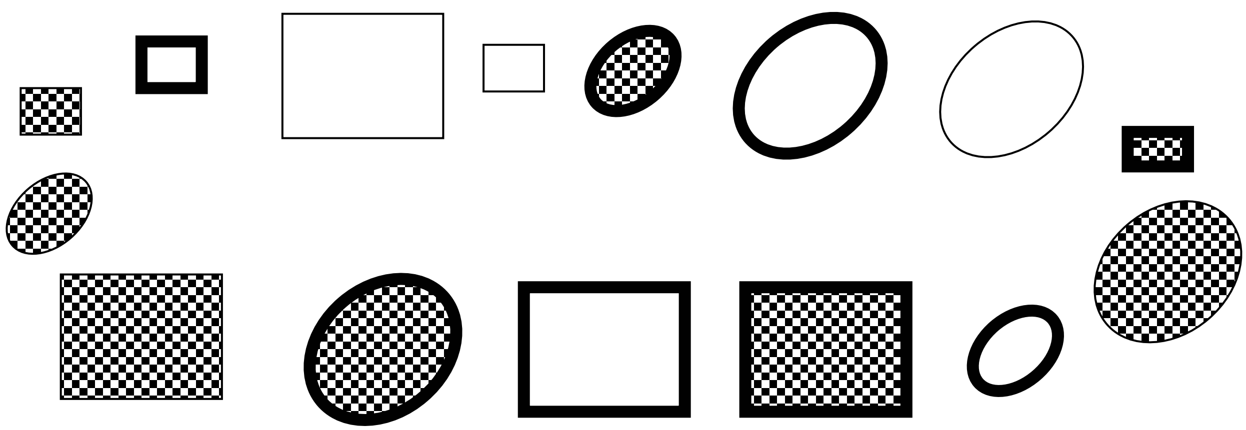
  

# Appendix 3.

## How to find colour sequences for two and for three colours

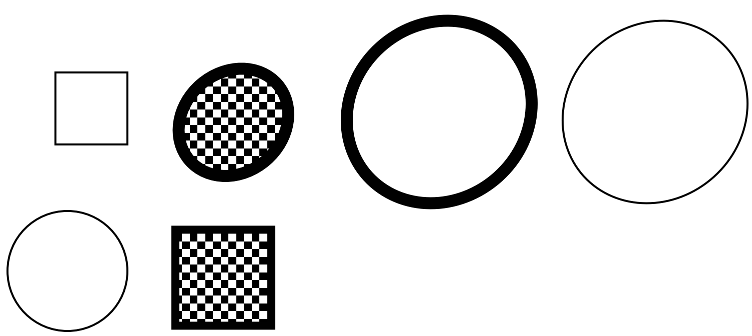
### 1. The four strip problem with two colours

From a set of attribute blocks (logic blocks) take all the squares and circles that are not blue. You will have the following collection of blocks:



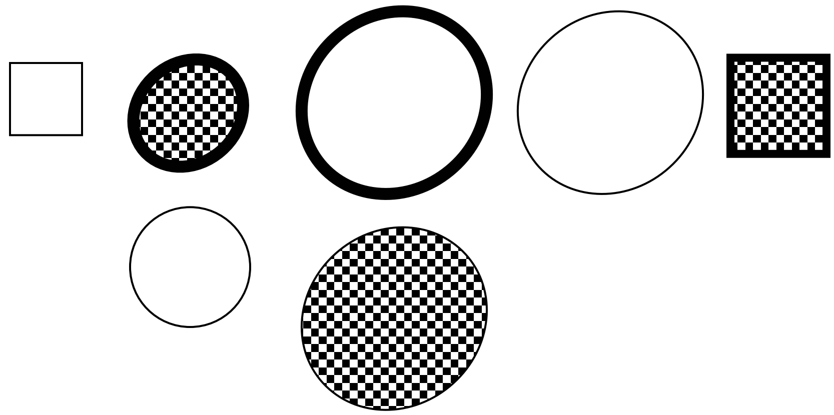
The blocks can be arranged in this position by doing the following:

Put the following four blocks out in a row and then put the small thin yellow circle next to the first one as shown below  and then see what should "go" with the second block in the same way as the small thin yellow circle "goes" with the first one.

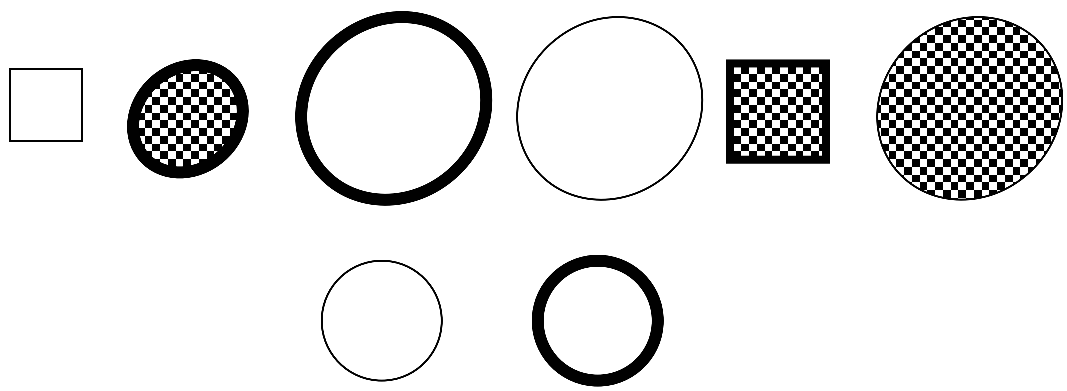


It will be the small thick red square. Put this block down as your fifth block and move the small thin yellow circle up to the second block. Which is the block that “goes” with the third block as the small  thin yellow circle “goes” with the second one?



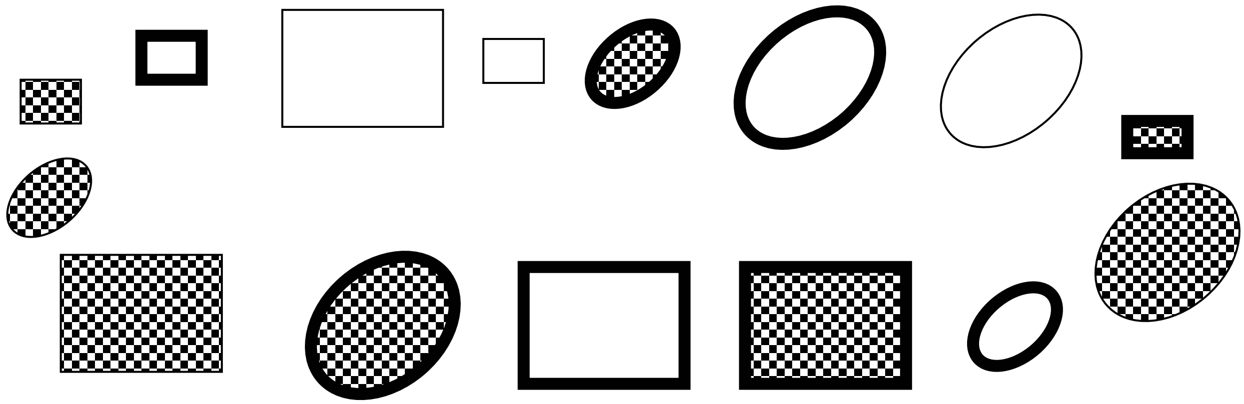
This will be the big thin red circle. So this will be the sixth block. Now move the small thin yellow circle next to the third block and find the block that “goes” with the fourth block in the same way as the small thin yellow circle “goes” with the third block.



This will be the small thick yellow circle. So this will be the seventh block.
Keep moving up the small thin yellow circle and continue the above procedure until all the blocks are used up. You will then get the cycle of blocks given at the start,

Now let us examine the cycle to see how the sizes, thicknesses, colours and shapes  change as we go round the cycle.



For your convenience, above is the cycle of blocks again.

If we look at the sizes, starting with the big thick red square and proceeding in the clockwise sense, we see the sizes change thus:

Big, big, big, big, small, small, small, big, small, small, big, big, small, big, small

Starting one block further back, for the thicknesses we obtain:

Thick, thick, thick, thick, thin, thin, thin, thick, thin, thin, thick, thick, thin, thick, thin

While starting with the big thick red circle, we get:

Red, red, red, red, yellow,yellow,yellow red, yellow,yellow, red, red, yellow, red, yellow

Starting with the small thin red square, the shapes change as follows:

Square, square, square, square, circle ,circle, circle ,square, circle ,circle, square, square, circle, square, circle.

So all the four attributes change according to the following rhythm

1 1 1 1 0 0 0 1 0 0 1 1 0 1 0

which is the sequence of colours for each strip if we wish to use four strips..

This is how the sliding of the strips would need to be done in order to fit with the given cycle of blocks:

```
 Y Y Y Y N N N Y N N Y Y N Y N Y Y Y Y N N N Y
Y Y Y Y N N N Y N N Y Y N Y N Y Y Y Y N N N Y
   Y Y Y Y N N N Y N N Y Y N Y N Y Y Y Y N N N
      Y Y Y Y N N N Y N N Y Y N Y N Y Y Y Y N N N
```

in the first row Y meaning big and N meaning small,
in the second row Y is thick and N is thin,
in the third row Y is red and N is yellow,
in the fourth row Y is square and N is circle.

Each column in the above arrangement is a code for a particular block.

The rules for "guessing" the next block round seem to be the following:

A thick one is always followed by a big one, a thin one by a small one.

If a block has one or three of the properties: big, thick, square, then the next block is thick. If it has two or none, the next block is thin.

If a block differs in an odd number of properties from the small thin yellow circle, then the next block is red. For an even number the next block is yellow.

If a block has one or three of the following properties: big, thick, red, the next block is a square, but if it has two or none, then the next block is a circle.

Now try to make up another sequence of fifteen blocks, starting with another set of four starting blocks. The block that is left out of the sequence does not have to be the small thin yellow circle, it can be any of the blocks. When you have constructed your cycle of fifteen, arrange your four strips, so that each column represents a block in the sequence and that the columns come in the same order as the blocks in the cycle. Then try to find the rule for "guessing" the block next to any given block. You might also try to find the rule for the preceding block, or the block two or three blocks away. There will be rules for all these!

Your choice of the first four blocks could be unfortunate, as for some choices you will get back to the first block before using up all the fifteen blocks. In such a case, just try another choice of four starting blocks. You might find it interesting to find out how to choose your first four blocks so that you inevitably use up all the blocks.

## 2. Colour sequences for five strips using two colours.

One way to access possible colour sequences for five strips is to use all the non-blue blocks and split the shapes in the following way:

Squares and rectangles are **RECTANGULAR**
Circles and triangles are **NON-RECTANGULAR**

Squares and circles have **HIGH SYMMETRY** (more then 3 lines)
Rectangles and triangles have **LOW SYMMETRY** (2 or 3 lines)

If we denote high symmetry, rectangular, big, thick, red and square by means of 1,
and low symmetry, non-rectangular, small, thin and circle by means of 0
we can establish a code for each block, by deciding, for example, that we shall refer to the properties of any block in the order:

symmetry, rectangularity, size, thickness, colour .

The small thin yellow triangle will represent the zero vector (0, 0, 0, 0, 0) as it has low symmetry and is non-rectangular. We can now choose five blocks, for example with codes:

1 1 1 1 1, 1 1 0 0 0, 1 0 0 0 0, 0 1 1 0 0, 1 1 1 0 1

which you can check are the following blocks:

the big thick red square, the small thin yellow square, the small thin yellow circle,
the big thin yellow rectangle, the big thin red square.

If you add the first, the third, the fourth and the fifth of the above vectors you will obtain the vector 1 1 1 1 0 as your sixth vector. Then if you add the first the third the fourth and fifth of the last five (including this sixth one you have just added!), then you will get the seventh verctor 1 0 1 1 1. If you carry on like this, you will find that your $32^{nd}$ vector is again 1 1 1 1 1. . You can then make five strips with the repeating pattern:

1 1 1 1 1 0 1 1 1 0 0 0 1 0 1 0 1 1 0 1 0 0 0 0 1 1 0 0 1 0 0

which you will notice comes up for all five attributes we are considering.

For your convenience, here is the sequence of vectors you will obtain:

1 1 1 1 1
1 1 0 0 0
1 0 0 0 0
0 1 1 0 0
1 1 1 0 1
1 1 1 1 0
1 0 1 1 1
0 0 1 0 0
0 0 0 0 1
0 1 1 1 1
1 0 1 0 0
0 1 1 0 1
1 0 0 1 0
0 1 0 1 0
1 1 0 1 0
1 0 1 1 0
0 1 0 1 1
1 0 1 0 1
0 0 0 1 0
0 0 1 1 0
0 0 1 1 1
0 1 0 0 0
1 1 1 0 0
1 0 0 0 1
0 0 0 1 1
0 1 0 0 1
1 0 0 1 1
0 0 1 0 1
0 1 1 1 0
1 1 0 1 1
1 1 0 0 1

so you can verify that the given cycle is indeed present in all the components of the vectors. You will be able to work out how to slide the five strips so as to get precisely the above sequence of columns.

Sometimes you can get another sequence, which is as follows:

1 1 1 1 1 0 0 1 0 0 1 1 0 0 0 0 1 0 1 1 0 1 0 1 0 0 0 1 1 1 0

It can be proved mathematically that no other sequences are possible.

# 3. Colour sequences for three colours.

Let the three colours be yellow, blue and red and let them be symbolized by means of the numbers 0, 1 and 2 respectively. The adding and multiplying rules of mod 3 arithmetic will apply. So it is to be remembered that $2 + 2 = 1$ and $2 \times 2 = 1$, and $1 + 2 = 2 + 1 = 0$

Also remember that 1 and 2 are inverses of each other and 0 is its own inverse.

So start with two two-colour "towers" which are not inverses of each other and add them to get the third "tower". Then add the second and the third "towers" to get the fourth tower. Carry on like this until you get back to your first "tower". Going round the cycle twice, you might get the following sequence:

```
1  2  0  2  2  1  0  1  1  2  0  2  2  1  0  1
1  1` 2  0  2  2  1  0  1  1  2  0  2  2  1  0
```

and it is easy to detect the repeating cycle:

```
1  2  0  2  2  1  0  1
```

and to see that the lower strip has been slid along to the right by one notch.

To "guess" the next upper colour from any given "tower", all we have to do is to add the upper and the lower colours.
The lower colour of the next "tower" can be predicted to be the upper colour of the "tower" we are looking at.

Now let us see what happens when we want to work with three colours and three strips. How can be find out what to write on the strips?

The "towers" will now have three colours: an upper one, a middle one and a lower one.
The arithmetical rules will stay the same.

First choose two "towers" which are not inverses of each other. Then choose a third "tower" which is not the inverse of either of the first two, nor the sum of the first two "towers", nor the inverse of such a sum, nor the sum of one of the two "towers" and the inverse of the other. Having done this we can now add all the three "towers" to get the fourth one. Then add the second, the third and the fourth to get the fifth one. Continue like this, always adding the last three towers to get the next one, until you have a cycle of thirteen. You might get something like the following:

```
1 0 1 2 0 0 2 2 1 2 2 2 0 1 0 1 2 0 0 2 2 1 2 2 2 0
2 1 1 1 0 2 0 2 1 0 0 1 1 2 1 1 1 0 2 0 2 1 0 0 1 1
0 1 1 2 1 1 1 0 2 0 2 1 0 0 1 1 2 1 1 1 0 2 0 2 1 0
```

Unfortunately this only gives us thirteen out of the possible twenty-six non-zero “towers”. The way to get in all of them is to take the inverse of every other “tower”,

So we shall get something like this:

```
1 0 1 1 0 0 2 1 1 1 2 1 0 2 0 2 2 0 0 1 2 2 2 1 2 0
2 2 1 2 0 1 0 1 1 0 0 2 1 1 1 2 1 0 2 0 2 2 0 0 1 2
0 2 1 1`1 2 1 0 2 0 2 2 0 0 1 2 2 2 1 2 0 1 0 1 1 0
```

It is easy to see that the sequence of colours to be put on the strips is for example the first row and the sliding must be done like this:

```
          10110021112102022001222120
               10110021112102022001222120
     10110021112102022001222120
```

If you add the upper colour to the middle colour and take the inverse of the sum, you will get the upper colour of the next “tower”.
If you add the middle colour to the lower colour, you will get the middle colour of the next “tower”
If you add the upper and the lower colours and take the inverse of the sum, you will get the lower colour of the next “tower”.

So this is one way of finding a sequence of colours that will work for three strips.

# Appendix 4

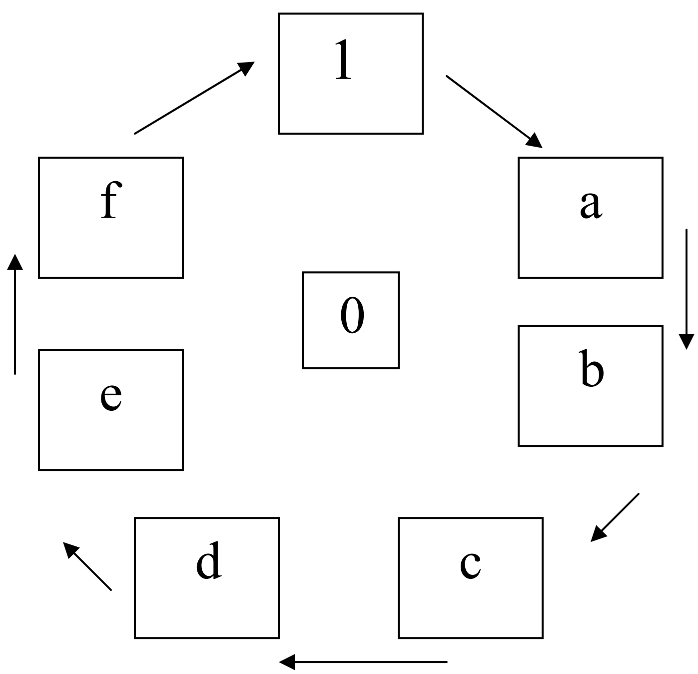
## Strips for a two dimensional space based on the eight and the nine element fields.

In the seven day cycle of fruit salads given in “The secrets of the circular villages, there was a different salad for each day of the week. This helped to define an “addition” of the days of the week. There was also a “bedtime treat”, which was obtained by adding a day to itself. I shall try now to proceed by using suitable symbols.

Let us symbolize the bed time treat by the number zero, and the days of the week as follows:

Sunday = 1, Monday = a, Tuesday = b, Wednesday = c, Thursday = d, Friday = e, and finally Saturday = f.

It is easy to keep in mind the addition rules for these symbols by looking at the cycle:




For any set of three of the above symbols in which two follow one another in the sense of the arrow and the third one is two arrows on, the sum of any two of the three will be the third. For example the set { a, b, d } satisfies these conditions and so we have

a + b = d, as well as a + d = b as well as b + d = a

The addition can be carried out in either order, namely a + b = b + a and so on.

Moving round the cycle can be considered to be a multiplication.

Multiplying by 1 would leave the multiplicand unaltered, multiplying by a would move the multiplicand through one arrow, by b through two arrows, by c through c arrows, by d through four arrows, by e through five arrows and by f through six arrows.
Multplying by f can also be considered as moving "back" one arrow, by e moving "back" two arrows, by d moving "back" three arrows.

Adding 0 will have no effect, while multiplying by zero will obviously yield zero.

Now we are ready to make our first strip.

Start with any two symbols except two zeros.
Multiply the first by c, the second by b and then add these products. This will be your third symbol on your strip Do not forget that multiplying by c means moving up three notches in the cycle, and multiplying by b moving up two notches in the cycle.

Now multiply the second symbol by c, and the third symbol by b and add these products. This will be your fourth symbol on your strip.

Carry on like this c-times-ing the last but one and b-times-ing the last and adding, to get the next symbol.

Carry on until you have 63 symbols in a row.
Then make another strip exactly like the one you have just made.
You can now slide these strips parallel to each other, but don't stop if you get two zeros in a column.

If you start with the two strips placed one under the other, forming a rectangle and then you slide the lower strip three notches to the right, you will get a sequence of columns with the following succession rule:

If you add the top one to c times the bottom one, you will get the symbol for the top of the next column.

If you add a times the top one to f times the bottom one, you will get the symbol for the bottom of the next column.

When you have carried out this operation 63 times, you will get back to your first column.

More concisely:

The top plus the bottom moved up three will give you the top next

The top moved up one plus the bottom moved back one will give you the bottom next.

Or “algebraically”:

top + c * bottom = top next [ 1→ a → b → c → d → e → f → 1 ]

a * top + f * bottom = bottom next

Here is how the strips might look:

```
0 c e b b 1 c b c 0 f a e e c f e f 0 b d a a f b a b 0
      0 c e b b 1 c b c 0 f a e e c f e f 0 b d a a f b a b
```

and you can check the rules I have mentioned,

You might like to check the following rule for getting the column after next:

e * top + e * bottom = top next but one

c * top + bottom = bottom next but one.

Of course I have “cheated”, as I simply multiplied the previous matrix by itslf!

You could now try your hand at finding the rule for the previous column.

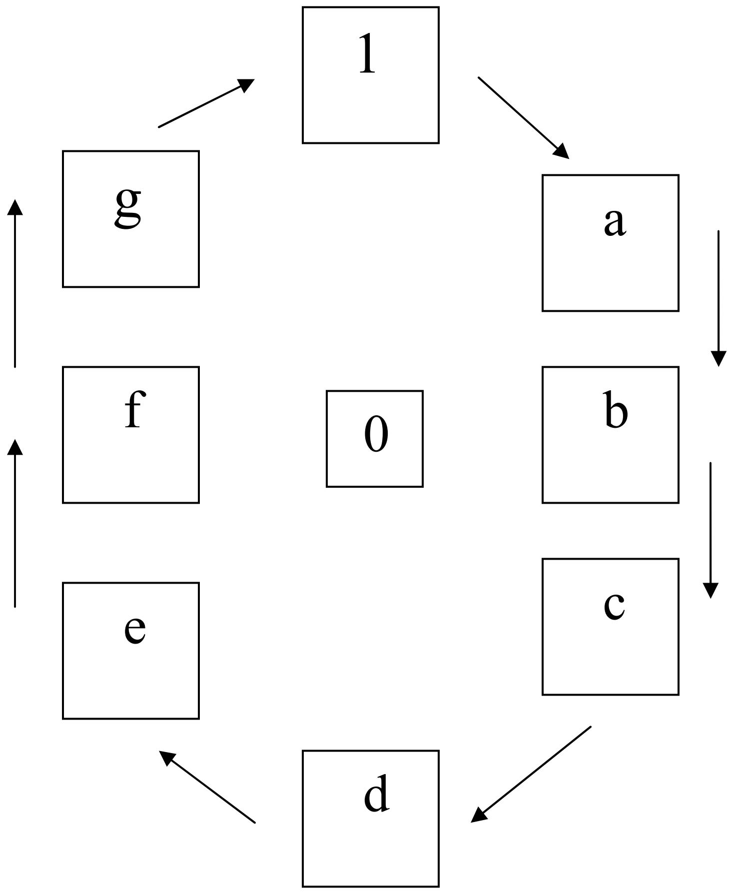
It is perhaps worth mentioning that from any two consecutive columns you can work out the next column by c-times-ing the first column and adding it to the b-times-d second column. This follows from the rule we used for constructing the strips.

I will leave it to you, as an interesting exercise, how you would make a succession of symbols if you wanted to use three or more strips!

You might also want to have a go at the problem in the case of the three by three field!
If you do not want to tackle it yourself, then turn to the next page.

There will of course be a cycle of eight for the multiplication, where the position of each letter in the alphabet will determine by how many arrows on you must advance to find the product.




The addition rules will be quite different. They will be the following:

An element added to itself will result in an element four arrows on.
The sum of two consecutive elements will be the element immediately following the second one.
The sum of two elements with an element in between them will be the element just before the first one of the two.
You obtain the sum of two elements with two elements in between by counting up to the third one after the second one of the two.
The sum of two elements with three elements in between them will be zero.

So the above rules define the nine-element field,
In a two dimensional vector space whose basis field is this field, there will be 9 x 9 vectors. So in order to create a cycle taking in all these vectors, less the zero vector, we need to find a two by two matrix, using the elements of this field, which is of order 80.

There are several such that I can think of.. For example:

e * top + e * bottom = top next

e * top + c * bottom = bottom next

which will yield something like the following

```
1 a 1 f b b e 0 f 1 e f e c g g b 0 c e
1 b g 1 g e a a d 0 e g d e d b f f a 0
```

To get a column ten notches after a given column, we multiply top and bottom by e (or move forward 5) so it will take eight lots such ten notches to get back to the first column, so the order of the matrix is clearly 80. But it is not easy to see where exactly the second strip would kick in, so we could do a more modest sliding with the initial part of the sequence and then find the matrix that produces that sequence of columns.

So let us try this:

```
1 a 1 f b b e 0 f 1 e f e c g g b 0 c e
    1 a 1 f b b e 0 f 1 e f e c g g b 0 c e
```

The matrix for the above seems to be:

e * top + d * bottom = top next

f * top + c * bottom = bottom next

The above strips appear to have quite a simple rule of construction. It is this:

After any two consecutive symbols you will always get "a" times the first symbol plus "b" times the second symbol as the one that follows the two you have picked. This means, of course, that out of any two consecutive columns you can calculate the next column by multiplying the first one by "a", the second one by "b" and adding these.

The field of sixteen elements would seem to be the next "interesting" field to try.

In the sixteen field we would have a fifteen cycle for our multiplication.

Let it be the following:

1 → a → b → c → d → e → f → g → h → i → j → k → l → m → n → 1

denoting the zero conventionally with the symbol 0.

The addition rules should be the following:

Any element added to itself will give 0

There will be three ways of making up sets of three elements such as any two of the three will yield the third one as the sum. These are the three ways:

(i) two consecutive letters, along with the third letter after the second of the two, will form a set of three. This means two consecutive letters, a gap of two letters and the one that follows will be the third.

(ii) the three letters are symmetrically distributed around the cycle. This means that between any two of the three letters there will be a gap of four letters.

(iii) take two letters with just one letter in between them, and then the sixth letter after the second of the two. This means that there will be a gap of one letter between the first two and a gap of five letters between the second and the third.

Multiplication id determined by the position of the letter in the alphabet. The 1 is the neutral, "a" makes a move up of one in the alphabet, "b" makes takes you on two letters and so on.

In a two dimensional space whose basis field is this sixteen element field will have 256 vectors, or 255 = 15 * 17 vectors, not counting the zero vector.

One matrix that will encompass all these 255 vectors one after the other is the following:

J * top + e * bottom = next top

F * top + k * bottom = next bottom.

Starting with 11, the first seventeen vectors obtained by this matrix are: (in capitals for clarity)

11, 1A, GD, KF, A0, KG, DF, JD, FD, CK, L1, ME, A1, CH, 0N, DJ, CG,

the next one being aa. So applying the matrix again we shall get sets of seventeen, starting with aa, then with bb, then with cc and so on So the matrix given is of order 255.

It is interesting to observe that each vector can be obtained from the two preceding ones by moving the first on one notch and moving the second one back one notch and adding them. Algebraically:

$$a * V_n + n * V_{n+1} = V_{n+2}$$

So the strips can be constructed from the above property.
Start with any two letters, and keep adding the first letter moved up one to the second letter moved back one of the last two letters you have found.

When you have made two such strips, you can find another matrix of order 255 by sliding not too far along the other strip. For example by sliding the lower strip two notches to the right starting from a position when the two strips are in the same position,  I believe you will get a sequence of vectors generated by the matrix:

m * top  +  c * bottom  =  top next

a * top  +  b * bottom  =  bottom next

```
1  A  D  F  0  G  F  D  D  K  1  E  1  H  N  J  G
      1  A  D  F  0  G  F  D  D  K  1  E  1  H  N  J  G
```