The theory of the six stages of learning with integers

(Published in "*Mathematics in Schools*", Volume 29, Number 2, March 2000)

Stage 1

Free interaction

In the case of the study of integers, this first stage will already have been experienced, as all that is necessary is to be aware that out of two kinds of objects sometimes there are more of one kind than of the other, some times less, and sometimes there are as many of one kind as of the other. Most people will have also experienced "counting" these differences, and will be aware that there are two things that they look for in the differences:

(i) Out of which of the two kinds are there more and which less?

(ii) How many more or less ?

This is probably done by siblings arguing with each other about one of them having more or less candy than the other, or going to bed later or earlier than the other or through any such simple "more or less" situation which we all encounter very early in life.

Stage 2.

Playing by the rules.

This is the stage in which we discover some rules and learn to play by them . Alternatively we could also invent our own rules and play by them, as we shall see in the examples that follow. This stage could be called: learning to play a game with rules.

(i) The dance story.

People coming to a certain place of entertainment have to obey the following rules:

(a) Everyone must enter or leave by going through the refreshment room.
(b) Anyone entering the refreshment room from the outside must choose a partner of the opposite sex, if there is any such person in the refreshment room. If there is no such person, he or she must wait until one turns up.
(c) You are only allowed to dance with a person of the opposite sex.
(d) Everyone must dance, while the music is playing, if there is a person of the opposite sex that they can dance with.

The dance hall leads off the refreshment room and is very big, and there are always lots of couples dancing, as dancing is that town's favorite pastime.

Now here are some questions.

(Question 1) There are 3 girls and a boy in the refreshment room (1 boy and 1 girl are just going into the dance hall). Then 2 boys and 1 girl come in from the outside. When the dancers have gone into the dance hall, who is left waiting in the refreshment room?

Answer. 1 girl is left waiting.

There were 2 less boys than girls at first, 1 more boy than girls came in, and we were left with 1 less boy than girls (zero boys being 1 less than 1 girl).

More briefly described we could say:

2 less   “added to”   1 more   results in   1 less

(Question 2)

There are 5 boys and 2 girls in the refreshment room (of course 2 boys and 2 girls are about to go dancing). Suddenly 3 girls are called home. Who is left in the refreshment room.

Answer: 6 boys.

The 3 girls might all have been dancing in the dance hall. When they left, the 3 boys, their partners, joined the other three boys in the refreshment room, leaving 6 boys there.

The 2 girls who were about to enter the dance hall might have been two of the three who had to go, leaving 5 boys in the refreshment room. The third girl will have been in the dance hall, and when she left, her partner, would have joined the 5 boys in the refreshment room, again making 6 boys.

3 more boys than girls   minus   3 less boys than girls

leave behind   6 more boys than girls

or 3 more   minus   3 less   leave   6 more

The manager of the place of entertainment has a large family, lots of boys and lots of girls. At times when few people are ordering in the refreshment room, he sends one of the children to knock on the window of the refreshment room. This means that everyone must leave. But it also means that

(i) if the “knocker” is a boy, then everyone must send back a person of the same sex,
(ii) if the “knocker” is a girl, then everyone must send back a person of the opposite sex.

If several people go and knock on the window at the same time, then each person leaving must send back as many persons as the number of people who came to the window, of the same sex for each boy at the window, and of the opposite sex for each girl at the window.

(Question 3)

There are 3 girls in the refreshment room. Then 2 girls from the manager's family come and knock on the window. When the "replacement customers" arrive, who will be in the refreshment room?

Answer. 6 boys.

This is because each girl who leaves must send back 2 boys. So the 3 girls together will send back 6 boys.

3 less boys than girls   window   2 less boys than girls

result in   6 more boys than girls

If we call "window-ing" "multiplying", then we have

3 less   times   2 less   equals   6 more

It is not hard to work out that by the "window rules" we shall have

more x more = more,   more x less = less

less x more = less,   less x less = more.

(ii) Walking East and walking West.

Johnny is walking up and down the road to exercise himself. The road runs in an East-West direction. He walks a certain distance East, then turns round and walks back in a Westerly direction, then again turns East and so on, until he gets tired. At the end of his walk he stops and calls his father to pick him up in the car, as he is too tired to walk home!

He will either finish East of his starting point or West of his starting point, or maybe he will finish exactly at his starting point.

(Question 1)

Johnny walks 1 km East and then 3 km West and has a rest. He thinks he can do another walk and so now does 2 km East and 1 km West. Does he end up to the East or to the West of his starting point and how far?

Answer. He ends up 1 km to the West.

We can write the two walks like this:

2 less East than West followed by 1 more East than West

is the same as 1 less East than West

or 2 less then 1 more is the same as 1 less

So you see that you can "add" Johnny's walks and always obtain another walk. But what about "subtracting" walks? You might just want to "subtract" a walk from one of Johnny's walks that he never actually made on that particular walk. Anyhow, what does it mean to "subtract a walk from another walk" ?

We could ask what part of the walk Johnny had done without having done a certain portion of it? For example:

(Question 2)

10 km East, 7 km West - 3 km East, 2 km West

would mean: What would Johnny have done if instead of 10 km East he had done 3 km less towards the East, and instead of 7 km West he would have done 2 km less towards the West ? He would have done 7 km East and 5 km West. So

Answer: 3 more - 1 more = 2 more.

But suppose we wanted to know what Johnny would have done had he not done 3 km East and 8 km West? He never did 8 km West, so it seems that the problem has no sense.

In order to make the problem “soluble”, we can introduce the notion of

EQUIVALENT WALKS

and say that in this walking game, any walk can be replaced by any other EQUIVALENT walk.

Two walks are equivalent if when they both start at the same spot, they also end at the same spot.

In our example we can replace Johnny’s walk by 12 East, 9 West. We can now “subtract” 3 East, 8 West and end up with 9 East, 1 West. So this “subtraction” can be expressed as

3 more - 5 less = 8 more

Johnny and his sister Mary both go out walking. But this time they just go for short walks in their back yard. But before going out, they throw a die. Three of the faces of the die are painted red, the other three green. The red faces have the numbers 1, 2 and 3 written on them (one number on each face), and the same is true for the green faces.

They decide that for every meter that Johnny walks, Mary has to do as many meters as what was shown on the die when they had cast it. If it had come up green, then Mary would always go in the same direction as Johnny, but if it it had come out red, Mary would always have to walk in the opposite direction to the one which Johnny was taking. They both walk on a little path which runs from East to West.

(Question 3)

Johnny walks 3 m East, 5 m West, and 6 m East.

The die shows a red 2. What does Mary do?

Answer. Mary will do 6 m West, 10 m East, 12 m West

or 4 more East red 2 yields 8 less East

You could throw the die more than once and either carry out the "commands" one after the other, or "count" whether there will be more green than red or less green then red in your total of the die-numbers. You can then count the "more green than red" as "same direction commands" and the "less green than red" as "opposite direction commands".

If "more" means "more East than West" and "less" means "less East than West", and "more" means also "more green than red" and "less" means also "less green than red", we can easily verify that we have the rules:

More or less East than West steps	More or less green than red die numbers	More or less East than West steps
more	more	more
more	less	less
less	more	less
less	less	more

The "rules of the game" are now well defined and we can add, subtract and "multiply" walks.

(iii) The circles and squares rules.

You must have a whole pile of counters available, some should be circular and some square shaped counters.

A circle and a square together is called a "zero pair".

Two piles of counters are EQUIVALENT to each other, if one can be obtained from the other by adding and/or removing some zero pairs.

Two piles of counters equivalent to each other can be exchanged with each other in what follows.

A pile in which there are no zero piles is said to be in standard form. A pile in standard form has only circles in it, or it has only squares in it.

Rules for adding.

We add two piles by simply putting the two piles together to make a united pile.

Rules for subtracting.

We subtract a pile from another pile by simply removing the pile to be "subtracted" from the pile from which it is to be "subtracted". If the pile "to be subtracted" cannot be found in the pile from which it is to be subtracted, we simply add enough zero pairs to the latter pile until the "subtraction" becomes possible.

Multiplication is done in the following way:

Make a pile A. This is the "multiplicand pile". Then make a pile B. This is the "multiplier pile". Then make a third pile, the "product pile" or the pile C, in the following way:

For every counter in pile A put as many counters of the same shape in pile C as there are circles in pile B. Also for every counter in pile A put as many counters of the opposite shape in pile C as there are squares in pile B .

We can refer to piles in which there are more circles than squares as "more piles" and piles in which there are less circles than squares as "less piles".

In what follows we shall refer to circles briefly by writing the letter C and to squares by writing the letter S.

(Question 1) Do we get a "more pile" or a "less pile" if we add

CCCCCSSSS and CCSSSSSSS ?

Answer: The first pile is a "1 more C pile", the second pile is a "5 less C pile". Putting them together we get a "4 less C pile".

(Question 2) From the pile CCCSSSSSSS take away the pile CCCCCS. What kind of pile do you get ?

Answer: we have to add two zero pairs to the first pile before we can remove a pile like the second pile from it. So the pile EQUIVALENT to the first pile that we have to consider is

CCC CS CS SSSSSSS

We can now remove CCCCCS and get SSSSSSSS

We have done the following:

(4 less) - (4 more) = (8 less)

(Question 3)

Into pile A (multiplicand pile) let us put S S S S C

Into pile B (multiplier pile) let us put S S C

The product pile will be:

CCCCS CCCCS SSSSC

which is a "3 more C pile", so we have the "multiplication":

(3 less) X (1 less) = (3 more)

We have now given three different activities for essentially the "same thing". In what way they are the "same thing" will be clearer in Stage 3.

Stage 3.
The comparison stage.

The three activities have been given precisely in order that only what is common to all three should be eventually retained. In order to see the common elements of the activities, we must see what corresponds to what as between activities. If this is not already quite obvious, here is a "dictionary", which "translates" one activity into one of the others.

Dance	Walks	Counters
boys	steps East	circles
girls	steps West	squares
people present at the dance	set of steps East and/or West	pile of circles and squares
only boys in refreshment room	more East steps than West steps	more circles than squares
only girls in refreshment room	less East steps than West steps	less circles than squares
people arrive in refreshment room	do one walk and then do another	put two piles of counters together
people leave the place of entertainment	find the walk that would have been done without a part of the walk	From a pile remove a pile that is in it
n boys knock on the window	n more green die numbers than red die numbers	n circles more than squares in the multiplier pile
n girls knock on the window	n less green die numbers than red die numbers	n circles less than squares in the multiplier pile

In each activity we are concerned with the difference between the number of one kind of “element” and the number of another kind of “element”.

In the dance these elements are boys and girls, in the walks they are displacement units East and West and green and red die numbers, in the case of the counters, they are circles and squares.

In each case we decide which of the two kinds we “count” . In the dance activity, we “count” the boys, since we say “more boys than girls” and “less boys than girls” and not “less girls than boys” and “more girls than boys”. It is immaterial which of the two kinds is chosen to be “counted”, but a convention must be made, so that we know which are “more” situations and which are “less” situations. In the case of the walks we “count” the East displacements, in the case of the counters we “count” the circles as “elements of reference”.

Stage 4.

Representation.

We have seen how the three activities are similar. It should therefore be possible to represent them all on the same figure. Below is such a suggested representation:

```
-9 -8 -7 -6 -5 -4 -3 -2 -1  0 +1 +2 +3 +4 +5 +6 +7 +8 +9

9,0   8,1   7,2   6,3   5,4   4,5   3,6   2,7   1,8   0,9
   8,0   7,1   6,2   5,3   4,4   3,5   2,6   1,7   0,8
      7,0   6,1   5,2   4,3   3,4   2,5   1,6   0,7
         6,0   5,1   4,2   3,3   2,4   1,5   0,6
            5,0   4,1   3,2   2,3   1,4   0,5
               4,0   3,1   2,2   1,3   0,4
                  3,0   2,1   1,2   0,3
                     2,0   1,1   0,2
                        1,0   0,1
                           0,0
```

The number after the comma refers to the elements which we “count”. For example (3,4) means that there is 1 more of the second kind than of the first kind.

The pairs with the same differences are always in the same column. At the head of each column a “more situation” is denoted with a plus sign and a “less situation” with a minus sign. The “as many as” situation is denoted by zero. In this way we obtain the NUMBER LINE, on which the “more” and the “less” situations are represented in order. Each “point” on the number line represents not just one situation but an infinite number of situations. What is common to all situations in the same column is that they have the same “moreness” or “lessness” property.

It is easy to see that if we perform an addition in any of our three activities, we shall be starting at a certain “point” and move to the right if we “add” a “more situation” and move to the left if we “add” a “less situation”

It is also easy to see that if instead of “adding” we “subtract”, then we move to the left if we subtract a “more situation” and we move to the right if we subtract a “less situation”.

Multiplication can be seen to be a kind of “inflation” of the distance of our starting point from the point representing zero. Multiplying by a “more” we remain on the same side of zero as our starting point, multiplying by a “less” we also inflate but also change “sides”.

If “less” “more” situations are regarded as “opposites” of each other, it is also easy to check that “subtracting “ will yield the same result as “adding the opposite”.

Clearly instead of “more” we could use its Latin equivalent, namely the word “plus”. Also instead of “less” we could use the Latin word “minus”, which means “less”.
In the next stage we shall see how the conventional symbol system can be developed by taking the above properties of the number line as starting points. It might also be useful to play about with some less conventional ways of symbolizing, to stop ourselves from “cheating” by referring to various rote learned algorithms which we remember from childhood!

Stage 5.

Symbolization.

We can now “play about” on the number line and develop a “language” to record what we find.

We could use the conventional plus and minus signs for adding and subtracting respectively, and use the plus and minus signs as suffixes when we wish to say “more” or “less”. For example to symbolize the adding of a “2 more” to a “3 less”, we could write

$$2_{+} + 3_{-} = 1_{-}$$

the equal sign meaning “comes to the same result as”.
The above is a “mathematical sentence”. The subject of the sentence is the left hand side, the predicate is the equal sign together with the right hand side. Or, if you prefer, you could think of the left hand side as one subject, of the right hand side as another subject, and of the equal sign as the predicate, which says something about the two subjects, namely that they are equal.

We can use the lower case letters x, y, z, … to denote “points on the number line yet to be chosen”. If we write the same letter more than once in the same sentence, the same “point” must be picked each time that letter occurs. On the other hand, we are allowed to pick the same point for two different letters.

We must not forget that each “point” represents a whole lot of pairs whose elements differ from each other by the same amount. A whole class of such pairs having the same difference between its elements is called an INTEGER.

Let us do some investigations. Choose two integers, namely two points on the number line. Call them x and y respectively. It is easy to check that if we start from the “point” x and “add to it the point y”, we shall arrive at the same result as if we had taken the point y and then had “added to it the point x”. We have to remember that “adding a plus point” means moving to the right, and “adding a minus point” means moving to the left.

If we are convinced of the truth of the above, we can write

$$x + y = y + x$$

We could use another method of writing the above. An upper case S could mean “The sum of” as long as there are two integers following whose sum we can be talking about. So the above finding could also be written as

$$S\,x\,y = S\,y\,x$$

If we take an “as many as” situation, represented by the “point” 0 and any other point x and “add them” , we shall have the point x as a result. This we can write as

$$0 + x = x \quad \text{or} \quad S\,0\,x = x$$

Now we can try something involving three “points” or integers. Let us call them x, y and z respectively. Suppose we “add” x and y and to the “sum” we “add” the point z. Do we arrive at the same final point if to the point x we “add” the sum of the points y and z? Whatever points you choose for x, y and z, you will find that the two ways of grouping and adding the integers involved always results in the same final integer. So we can write

$$(x + y) + z = x + (y + z) \quad \text{or} \quad S\,S\,x\,y\,z = S\,x\,S\,y\,z$$

where the bracket convention is the usual one, namely that any operations within brackets must be carried out before any operations outside them are performed.

In the “subject” S S x y z it should be noted that the second S refers to adding x and y, while the first S refers to adding S x y and z. In the “subject” S x S y z the second S tells us to add y and z, while the first S tells us to add x and S y z. Note that in the second symbol system we have no need of brackets.

We can arrive at corresponding properties for our multiplication, by examining what happens on the number line. We can use upper case P for "the product of", and so "discover" the following "rules of the integer game", using the star as a symbol for multiplying:

$$x * y = y * x, \quad 1 * x = x, \quad (x * y) * z = x * (y * z) \text{ or}$$

$$P\,x\,y = P\,y\,x, \quad P\,1\,x = x, \quad P\,P\,x\,y\,z = P\,x\,P\,y\,z$$

There is also an interesting property in our "integer game" which connects the two operations, namely that of "adding" and that of "multiplying". It is not hard to see that for any three integers x, y and z we have

$$(x + y) * z = (x*z) + (y*z) \text{ and } x * (y + z) = (x * y) + (x * z)$$

$$\text{or} \quad P\,S\,x\,y\,z = S\,P\,x\,z\,S\,y\,z \text{ and } P\,x\,S\,y\,z = S\,P\,x\,y\,P\,x\,z$$

Adding and multiplying are BINARY OPERATORS. By this is meant that we need two integers before we can either add them or multiply them (bis being Latin for "twice").

We shall need another OPERATOR to express things precisely. We need a "reverser", namely an operator that turns a "more" into a "less" and a "less" into a "more". For the moment let us stick to our S, P symbol system, and denote our REVERSING OPERATOR by means of an upper case N. This means that, for example

$$N\,4_- = 4_+ \quad \text{or} \quad N\,3_+ = 3_-$$

and so on. "Reversing" a point x will send us on the opposite side of zero to the one in which we are, but at the same "distance" from the zero. Clearly, if we reverse twice in succession, we find ourselves where we were before. So one obvious property of our N operator is

$$N\,N\,x = x \quad \text{for any integer } x$$

It is also quite easy to see that if we add an integer to its "opposite", we shall arrive at the zero point. So we can write

$$S\,x\,N\,x = 0 \quad \text{for any integer } x$$

What about reversing the sum of two integers? Do we get the same result if we add the reversed integers? This is also easy to check on a few examples, and so we can write

N S x y = S N x N y for any two integers x and y

What happens if we reverse the product of two integers? Do we get the same result as if we take the product of the reversed integers? Unfortunately not so! But we do get the same if we multiply the first integer reversed by the second integer. This “discovery” can be written like this:

N P x y = P N x y for any two integers x and y

There are a number of other properties we can “discover”, and then symbolize, using our new language. For example

P 2 x = S x x, P 3 x = S S x x x, P N x N y = P x y

in fact there is clearly no end to the “description” of what we can do with our number line.

So there is a problem. How much of the above type of description is SUFFICIENT for describing EVERYTHING that can be done? This is a very vague question, so let me try to make it more precise:

Suppose we have written down a certain number of the properties of the number line. Then let us suppose that we have found some RULES through the application of which we can DERIVE some of the other properties that are seen to be true on the number line. How many properties and which ones do we have to write down, so that, using our RULES, we can arrive at ANY property that is verifiably true on the number line?

In the next section we shall make an attempt to give such a set of properties and such a set of rules.

The initial properties will be called AXIOMS.
The CHAIN, leading from the AXIOMS to some other property, using the rules, will be called a PROOF.

The property reached at the end of the chain, will be called a THEOREM.

We have seen that subtracting an integer comes to the same thing as adding its “opposite”, so if “taking the opposite” is taken care of by means of the N operation, there is no need for any explicit “subtraction” in the system.

Stage 6. The S and P game.

Formalization.

Game 1.

The “well formed formulas” or W’FF’s.

A WW’F is defined as follows:

(i) Any lower case letter by itself, or any numeral by itself is a W’FF.
(ii) N followed by a W’FF is a W’FF.
(iii) P followed by two W’FF’s is a W’FF.
(iv) S followed by two W’FF’s is a W’FF.

Numerals 1,2,3, … will be written instead of 1_+,2_+,3_+…for brevity's sake.
Negative numerals will be written N1, N2, N3, N4, …..
Here are some W’FF’s:

3, N x, P 3 4, S 4 N 3, P x S y N z, P P 2 3 4, P 2 P 3 4

To play this game each player grabs a handful of pieces. Each player then tries to make a W’FF as long as he or she is able to with the pieces grabbed. The longest W’FF is the winner. Each player can challenge the W’FFness of the opponent’s W’FF. A successful challenger then becomes the winner.

We shall assume that S 1 1 = 2, S 1 2 = 3, S 1 3 = 4, and in general S 1 n = n + 1.

Game 2.

Transforming a W’FF into another W’FF.

Substitution rule.

Any lower case letter in a W’FF an be replaced by any W’FF, as long as the same letter is always replaced by the same W’FF.

In what follows, x and y and z stand for any W’FF’s.
It is permitted to replace any combination on the left of one of the following “equations”, with the combination that stands on the right of that “equation”or vice versa.

(1) P 1 x = x, (2) P x y = P y x, (3) P P x y z = P x P y z

(4) S 0 x = x, (5) S x y = S y x, (6) S S x y z = S x S y z

(7) P S x y z = S P x z P y z, (8) P x S y z = S P x y P x z

We can write the equal sign between a W'FF and any other W'FF into which we have transformed it by the rules.

Here are some examples of transforming a W'FF into another W'FF:

Start with P S x 1 x
(obtained from the left of (7) by replacing y by 1 and z by x)
Now writing the right of (7) with these "values", we have
S P x x P 1 x, but by (1) P 1 x = x, so we have
S P x x x.
So we have transformed P S x 1 x into S P x x x.
So we can write P S x 1 x = S P x x x.
Let us connect S 2 2 to 4.
S 2 2 = S S 1 1 2 = S 1 S 1 2 = S 1 3 = 4
For the second step we use rule (6)
Here is a long chain connecting

S S P x x P 2 x 1 to P S x 1 S x 1

S S P x x P 2 x 1, now use 2 = S 1 1 and we get

S S P x x P S 1 1 x 1 and by rule (7) we have

S S P x x S P 1 x P 1 x 1 now by rule (1) we have

S S P x x S x x 1 and then using rule (6) we have

S S S P x x x x 1, now using rule (1) again we can write

S S S P x x P x 1 x 1, and now by using rule (8) we have

S S P x S x 1 x 1; we can use rule (5) and get

S S x P x S x 1 1 then by rule (6) we get

S x S P x S x 1 1   then again by (5)

S x S 1 P x S x 1 then again by (6)

S S x 1 P x S x 1   then by (1)

S P 1 S x 1 P x S x 1   then by (7)

P S 1 x S x 1 and finally by (5) we have

P S x 1 S x 1   so we have made the desired connection!

Let us see whether any of our rules can be "derived" from any of the others. For example, could we connect the left and the right hand sides of equation (8), by using the rules (1) to (7) ?

Let us try and start with   S P x y P x z.

Using rule (2) we can write

S P y x P z x, then by rule (7) we can write

P S y z x and again by rule 2 we can write

P x S y z, so we have made the connection.

Rule (8) is not INDEPENDENT of the other rules, as it can be DERIVED from them.

Are there any other rules listed that are DEPENDENT or DERIVABLE from the others ?

Try to make other "connections". Some are easier than others. For example we can easily "connect"

P 2 x   with   S x x

Start with P 2 x, then use the fact that 2 = S 1 1 and write

P S 1 1 x then by (7) we can write

S P 1 x P 1 x and then by (1) we get

S x x and we have made the connection.

Game 3.

The rules for working with N.

Here are the “connecting rules” for using with W’FF’s containing N:

(9) S x N x = 0
(10) N N x = x
(11) N S x y = S N x N y
(12) N P x y = P N x y

As an example let us try to connect P N x N y to P x y.

Start with P N x N y then using rule (10)

N N P N x N y then by rule (12)

N P N N x N y then by (10) we have

N P x N y, then by rule (2) we can write

N P N y x, then using rule (12) again we have

P N N y x and finally by rule (10) we can write

P y x but by rule (2) we can also write

P x y

So we can write P N x N y = P x y

It might be fun to start with P S x 1 S x N 1 and try to connect it to
S P x x N 1. How would you do it ? And how do I know that it can be done
?
Start with

P S x 1 S x N 1 by (7)

S P S x 1 x P S x 1 N 1 by (7) twice

S S P x x P 1 x S P x N 1 P 1 N 1 by (6)

S S S P x x P 1 x P x N 1 P 1 N 1

Before going on it would be good to be convinced that
P x N 1 and N x can be connected;

P x N 1 = P N 1 x = N P 1 x = N x (13)

and putting x = 1 we also have

P 1 N 1 = N 1 (14)

So we use (1) (13) and (14) and go on to

S S S P x x x N x N 1 then by (5)

S S S x P x x N x N 1 and by (5) again

S S N x S x P x x N 1 then by (6)

S S S N x x P x x N 1 and by (5) again

S S S x N x P x x N 1 so by (9)

S S 0 P x x N 1 and by (4)

S P x x N 1

and so we connected P S x 1 S x N 1 to S P x x N 1.

You have known all the time that S stands for “The sum of”, that P stands for “The product of” and that N stands for “The negative of”. But the "game" can be played without knowing this! But you must take care that S and P are followed by two well formed formulae and N by just one!

The “game” shows how complex the situation is if we wish to rely on absolutely formal methods. The traditional method of algebra teaching is neither intuitive nor rigorously formal, but falls between these two stools which are very far apart from each other. If we want to be intuitive, let us use all sorts of materials such as little square shapes, trays, cups and beans, colored counters or balance beams. If we wish to be “formal”, let us be really formal, and become aware of every formal step that allows us to go from one expression to another, supposedly algebraically identical with each other.

It is really salutary for a teacher to go through step by step, with a mathematical tooth comb, that which is required to establish formally that

$$(x + 1)^2 = x^2 + 2x + 1$$

or in our formal system that

P S x 1 S x 1 = S S P x x P 2 x 1

or that $x^2 - 1 = (x + 1)(x - 1)$ or

S P x x N 1 = P S x 1 S x N 1

The advantage of an unfamiliar system is that we cannot “slur” over anything. We must use the rules one by one in an orderly succession and so become aware of how much is taken for granted when the “usual” rules are applied and thought to be adequately “formal”.

One advantage of the "S and P system" is that parentheses are not required. Another advantage is that it follows closely how the operations referred to would be expressed in the "vernacular". For example:

P 3 S 4, 5 "means" "Find the product of 3 by the sum of 4 and 5".

Once the fine detail of the "proofs" has been appreciated and truly understood, students can safely return to the conventional ways of expressing mathematical statements as they are usually found in text books.

For more examples, see the web site at http://www.zoltandienes.com

