Bruce and Alice learn a logic game.

After Bruce and Alice had perfected their skills on all the five eight-games, they realized that there was a lot more to learn. One day at breakfast Kara said to Bruce:

"In your school where you usually live, do you learn any Logic?"

"Logic?", replied Bruce in a surprised tone, "I don't think so. I thought logic was just about speaking correctly. We do learn all about that. We call it Grammar."

"Oh no!", said Kara, "Grammar is only to do with how we speak, but Logic comes into everything we think about! It is about reasoning and drawing conclusions!" added Kara.

Okto was overhearing this conversation, and could not help joining in. He said:

"In Ruritania we learn nearly everything through playing games. If we showed Bruce and Alice some logic games, they would know what we were talking about"

"That would be fun", joined in Alice, "How many players do you need for a logic game?"

"We would need 10, or eight, as you would say", replied Okto, "for the simple ones".

"I might have known!", said Alice smiling, "I forgot! We are in Eightland!"

"Let's get all the others and play!" said Gono enthusiastically, "We can play at Logic as our maths lesson today, I'll go and ask the maths teacher."

With this Gono ran off and soon came back saying that it was quite in order to show Bruce and Alice some Logic games as their maths lesson for the day. Okto took over, as usual, and said:

"We shall need four rows of two chairs each. Each of us will have a chair to sit on. You can think of the chairs as a small bus, which will have a front half and a rear half, a left half and a right half. It will also have a middle half and an end half. If we think of the two chairs next to each other as the rows, the second and the third rows form the middle, the first and the fourth rows form the end half.. So each of us will have three names: either front or rear, then either left or right and finally either middle or end. So let us find our seats!"

The children sat in the chairs as follows

Left hand side

Bruce	Alice	Alo	Ata	→
Unta	Okto	Kara	Gono	

Right hand side

"What are the rules of the game?", asked Alice.

"We start either by everybody sitting, or by everybody standing. During the game, only those who are standing are counted as being there. There are three moves: ADD, KEEP and EXCHANGE. In each move you have to say one of the six words: front, rear, left, right, middle, end to follow the move-word."

"Those that you speak of after the move-word have to stand, do they?", asked Bruce.

"Yes", replied Okto, "For example if we say : ADD FRONT, then those in the rear stay as they are, but all the front people have to stand up. Any front person already standing just stays standing, any sitting front persons stand up"

"I suppose in KEEP FRONT", ventured Alice, "Only the front people stand. But do they all stand?"

"No", replied Okto, "Only those front people that were already standing are kept standing. You cannot keep what you do not have. And of course any rear people that might be standing, must sit down!"

"Suppose that Okto, Alo and Kara are the only ones standing", chimed in Ata, ", then on a KEEP FRONT move Okto would sit down but Alo and Kara would stay standing. Equally, Ata and Gono would keep sitting, as we cannot keep them, since we did not have them in the first place! Am I correct?"

"Absolutely", reassured her Okto. "Now let me explain the EXCHANGE move. For example EXCHANGE MIDDLE would mean that any standing middle ones must sit down and any sitting middle ones must stand up. But the end ones must stay as they are, as the move does not refer to them!"

"So you don't exchange middle ones for end ones!", said Unta, "you only exchange middle ones for middle ones! The exchange is not between middle and end, it is between sitting and standing!"

"Now the game is played in the following way", said Okto, "We decide on the persons whom we want to see standing at the end. Then we try to reach that situation in as few moves as possible. Players take it in turn to try and solve the same problem, and the one who solves it in the least number of moves is the winner."

"Let's finish with all the boys standing up, so we can sit comfortably!", suggested Unta.

They all agreed on this problem. They took quite a long time, but finally came up with a solution in three moves. Here is what they did, starting with all standing :

Bruce Alice Alo Ata Unta Okto Kara Gono	Keep left →	Bruce Alice Alo Ata sitting sitting sitting sitting
	Exchange front →	Bruce Alice sitting sitting Sitting sitting Kara Gono
	Exchange middle →	Bruce sitting Alo sitting sitting Okto sitting Gono

"Now let's get all the girls to stand!", suggested Bruce. "But let's start with everyone sitting this time!"

They soon found that if they started with ADD RIGHT, they could go on as before, following with an EXCHANGE FRONT, and finishing with an EXCHANGE MIDDLE!

"Let's try three persons", suggested Ata, "What about Bruce and Alice and Okto, starting with everyone standing?"

"How many moves do you reckon we shall need?", asked Okto.

"The only way to find out is to try", replied all the others almost at once. And try they did. Finally they managed to get their three chosen persons standing after only three moves. They did the following sequence:

KEEP LEFT, ADD MIDDLE, KEEP REAR

And only Bruce, Alice and Okto remained standing at the end of the third move.

"Now, what about getting those that finished standing to finish sitting and those that finished sitting to finish standing!", suggested Kara.

"Perhaps there is a way of knowing how to do this, by looking at the problem we have just solved", suggested Alice "But I really wouldn't know just how!"

"Well", said Bruce, "Let us solve the problem and compare the two solutions!"

This was agreed. It was also agreed to start with everyone sitting, since they were trying to solve the "opposite" problem! This is what they finally came up with:

ADD RIGHT KEEP END ADD FRONT

As the shortest way to finish with Unta, Alo, Ata, Kara and Gono remaining standing.

"I can see something!", said Alo, "between the two solutions KEEP and ADD are changed over, as well as RIGHT and LEFT, also MIDDLE and END, also FRONT and REAR!. Is that the secret way of solving an opposite problem?"

"Maybe not quite", said Okto, "because look at our first two problems. When we wanted the boys to stand we had

KEEP LEFT EXCHANGE FRONT EXCHANGE MIDDLE

And when we wanted the girls to stand we had to do

ADD RIGHT EXCHANGE FRONT EXCHANGE MIDDLE

Where only the first move changes!"

"Yes, but only the first move is an ADD or a KEEP!", objected Bruce "Maybe with the EXCHANGE moves nothing changes, but we have to change KEEP and ADD, together with the words we say after them!"

"Maybe you are right", said Okto, "Perhaps we should try another one, and see if what you say is true in another example. What about trying to keep Alice and Kara standing?"

The children agreed to solve that problem, as well as its "opposite", in which Alice and Kara should finish up sitting while the other six children should end up standing. They took quite a long time solving these two problems, but they did not want to give up, and they finally came up with the following solutions :

ALL STAND KEEP RIGHT EXCHANGE REAR KEEP MIDDLE

ALL SIT ADD LEFT EXCHANGE REAR ADD END

"Your explanation seems to be working", admitted Okto to Bruce "But perhaps we should do some more, as we might have just struck lucky with these examples!"

"What about Bruce, Alice and Kara?", suggested Unta.

The best they could do with that problem and its "opposite" was a four move solution. They tried for a long time until Bruce finally had an idea:

"Let's work backwards", he said "I wonder what the last move must be to get Bruce, Alice and Kara? It cannot be an ADD, as you would have to have all the children from one half of the bus, and we have only three not four. And it cannot be a KEEP, because if it were a KEEP, all the children who were standing up at the end would have to come from the same half of the bus, and they clearly do not. So at least we know that the last move must be an EXCHANGE!"

"That's pretty smart", admitted Okto, "but what would have been exchanged?"

"May be the REAR would have been exchanged, as then before the exchange, all the standing children would have been on the right. And that would mean that the move before the last should have been a KEEP RIGHT. This might have meant that the children sitting on the left of these would have been standing before that last move but one!"

"I see!", said Alice, " And we could get those six children by doing KEEP REAR then ADD MIDDLE, or by KEEP MIDDLE and ADD REAR.. So it seems to me, that the moves must be:

ALL STANDING KEEP REAR ADD MIDDLE KEEP RIGHT EXCHANGE REAR

And the moves for the other five to end up standing then must be

ALL SITTING ADD FRONT KEEP END ADD LEFT EXCHANGE REAR

Shall we go through both of these and see if they are correct?"

They did so, carefully, so as not to make any errors, and found Bruce and Alice's solutions to be correct. Then Alice asked, with a wry smile on her face:

"I bet you have another version of this game up your sleeves", addressing herself to Okto.

"You are dead right", replied Okto "We can use words or we can use numbers. Which version do you want?", said Okto, addressing himself to nobody in particular.

"Let's have the number one first!", suggested Bruce.

"Are you sure you can handle numbers expressed in base eight ?", asked Okto.

"Of course!", replied Bruce and Alice together "You just have to imagine that you do not have any little fingers. So eight fingers count as one person, nine fingers as one person and one finger and so on."

"All right, then", replied Okto, "The game is played with the factors of 36 , or thirty as Bruce and Alice would say.. What are the factors of 36, Alice ?"

"Let me see", said Alice, thinking hard, "Here you are: they are

1, 2, 3, 5, 6, 12, 17, 36

I believe, expressed in your number system!"

"Four of these are odd and four are even. You can easily tell, since our rule for that is the same as yours. Four of them are multiples of 3 and four are not. The numbers 3, 6, 17 and 36 are multiples of 3, the numbers 1, 2, 5 and 12 are not. Then four of our numbers are multiples of 5, and four are not. The numbers 5, 12, 17 and 36 are multiples of 5, while the numbers 1, 2, 3 and 6 are not.. The two digit numbers and of course the 5 are 5-numbers, all one digit numbers except the 5 are not; that is one way you can remember them!"

"How do you play with these numbers?", asked Alice.

"You write each one on a piece of paper and have a box ready into which to put them or from which to remove them. When all the numbers are in the box, that is like when all the children in the bus are standing. When there is nothing in the box, that is like all the children in the bus are sitting. The moves are the same. Adding is adding to the box, Keeping is keeping in the box, and exchanging is exchanging a certain kind of number between those that are in the box and those that are not"

"All right", said Alo, "Let's keep the numbers 1, 2 and 3 in the box, the rest finishing outside the box"

"All three are one kind of number", said Bruce.

"They seem very different to me", said Ata.

"They are all non-5-numbers", said Bruce, "Not one is a multiple of 5, do you see?"

"And the ones we want are either odd or non-3 numbers", ventured Alice.

"In that case, starting with all the numbers in the box, we should say

KEEP ODD then ADD NON-3 and then KEEP NON-5

Going through the following sets of numbers:

[1, 2, 3, 5, 6, 12, 17, 36] KEEP ODD [1, 3, 5, 17] ADD NON-3 [1, 2, 3, 5, 12, 17] then

KEEP NON-5 gives us [1, 2, 3]. Three moves, not bad eh?"

"Now let us get the set [5, 6, 12, 17, 36] in the box!", suggested Bruce.

"Will the same trick work as the one we used with the bus?", wondered Kara.

"Let's try", said Alice. "Let us start with no numbers in the box and change Keeps and Adds around and change the type of number as well to which each move refers!"

So they did the following :

[ ] ADD EVEN [2, 6, 12, 36] KEEP 3 [6, 36] ADD 5 [5, 6, 10, 17, 36]

and they were "home"

"Do you know one where we have to use the EXCHANGE move?", asked Unta.

"I think I do", replied Okto, "what about the set [1, 6, 17] ?"

"They are not all the same kind of number!", cried Bruce "That's how you knew, isn't it, Okto?"

"Yes, of course", replied Okto, "But there is a little surprise in the solution. See if you can solve the problem of getting just these three numbers in the box!"

They took a long time before they could solve this one. Although they knew that the last move had to be an EXCHANGE, they eventually decided that the last move but one could not in any way be a KEEP in this case, nor an ADD, so that the solution had to end with two EXCHANGE moves! This was, no doubt, Okto's surprise!. Here is what they came up with, solving the problem is five moves:

[All] KEEP NON-5 [1, 2, 3, 6] ADD 3 [1, 2, 3, 6, 17, 36] KEEP even [2, 6, 36]

then EXCHANGE 3 [2, 3, 17] EXCHANGE NON-5 [1, 6, 17]

In order to get the set [2, 3, 5, 10, 36] into the box, starting with nothing in the box they did:

[Nothing] ADD 5 [5, 12, 17, 36] KEEP NON-3 [5, 12] ADD ODD [1, 3, 5, 12, 17]

then EXCHANGE 3 [1, 5, 6, 12, 36] then EXCHANGE NON-5 [2, 3, 5, 12, 36]

"The trick still works" said Bruce, "Did you invent it, Okto", asked Bruce, turning to Okto.

"I am afraid I cannot claim that honor!", replied Okto, "There was a person whose name was De Morgan, who lived a long time ago, who invented it. I don't think, however, that he played the game in quite this kind of way!"

"How do you play the game with words?", asked Gono "Can it be played with any words?"

"It is better to play it with words or with sentences that can easily be split into halves in three different ways", replied Akto, "For example you could play the game with the sentences:

[I come, he comes, we come, they come, I came, he came, we came, they came]

We then have these ways of splitting our set of eight sentences into two sets of four sentences:

First and third person, singular and plural, present and past"

"I see", said Bruce, " We can then KEEP PRESENT, EXCHANGE SINGULAR, or even ADD THIRD PERSON, and so on! It would really be the same game. We would just have to decide for each problem which sentences we should have in our box in the end!"

"What I would like to know is what the hardest problem is!", said Alice "I suppose the hardest is the problem that needs the largest number of moves to solve it!"

"The [1, 2, 3] problem is the hardest of those problems that can be solved". Replied Okto "It cannot be done under five moves."

"Do you mean to say that some problems cannot be solved?, asked Kara.

"Try putting just two elements in the box that are different from each other in all three ways being considered. You will never succeed with the rules we have made." explained Okto, "Naturally if you wanted six elements such that the remaining two differed from one another in all three ways, you would also be unsuccessful."

"Is that the only impossible case?", asked Kara.

"No, there is another way of setting an impossible problem". said Okto, "If you want four elements, one of which is different from each one of the other three in just one way, you would also find yourself in the soup!"

"I know", said Bruce, "The set [1, 2, 3, 5] would be impossible. You see, 1 is different from 2 in 1 being odd and 2 being even. It is different from 3 by 1 not being a multiple of 3 and 3 being one. It is different from 5 in 1 not being a multiple of 5 and 5 being one. You could never have just the four smallest numbers in the box by themselves!"

"Would the remaining numbers, that is the set [6, 12, 17, 36] also be impossible to get?" asked Alice.

"The way you can find out", chimed in Bruce, "is by looking for a number amongst these four numbers that is different from each of the other three numbers in one way only!"

"Yes, of course, there must be one", said Kara pensively, "But which of the four is the one?"

"I know!", said Alice, "It must be the number 36. This number is different from 17 in that 36 is even and 17 is odd, they are both multiples of 3 and multiples of 5. So that is just one difference. Then 36 is different from 12 in that 12 is not a multiple of 3 and 36 is a multiple of 3, they are both even and they are both multiples of 5! So that again makes just one difference!"

"And the number 6", added Gono, " is not a multiple of 5, while 36 is a multiple of 5. They are both even and they are both multiples of 3, so that again makes just one difference!"

"So [6, 12, 17, 36] is also an impossible set, by our rule!", concluded Alo.

"What is all this to do with logic?", Alice ventured to ask, trying not to sound too ignorant.

"Let us look at the set of words:

[ban, bat, bin, bit, fan, fat, fin, fit]

and then try to find the largest set in which every t-word is an a-word", suggested Okto. "Can you tell me which words would make up that set of words?", said Okto, addressing himself to Alice.

"Would it be [bat, fat] ?", said Alice tentatively.

"Well, those two words would have to be in the set, but you could also have all the n-words in it. Our set, the largest possible, would be:

[ban, bat, bin, fan, fat, fin]

since it is still true that every t-word in that set is an a-word. We have said nothing about the n-words, so they can all be included, whether they are a-words or i-words", explained Koto.

Kara wanted to be a part of the "explanation" too, so she chimed in at this point:

"If you had all the words in the box, what moves would you have to carry out so as to have the set [ban, bat, bin, fan, fat, fin] in the box?", Kara demanded.

"That's easy", said Bruce. "KEEP a-words, then ADD n-words. That will take care of all the a-words, and out of the i-words only the n-words would get in the box"

"So, if I picked a t-word out of what is in the box, it would have to be also an a-word" added Kara as a very logical conclusion. "You see, logic is something to do with if-then, and things like that, isn't that so, Okto?", she added.

"That is quite right", replied Okto

"It seems also to be true", said Bruce, "that if I pick an i-word, it is bound to be an n-word!"

"You have hit on something interesting", said Okto, "Think of any two situations and call them X and Y, but make sure that every time X occurs, Y also occurs. Then it FOLLOWS that if Y does not occur, then X does not occur either!"

"Oh yes!", said Kara, "We were learning about all this at school the other day, I think the teacher called it CONTRAPOSITION. Is that right?"

"Let X mean NOT LEARNING ENOUGH and let Y mean "BEING IN PRISON", suggested Alice. "In this case, at any rate in Ruritania, , if X is so, then also Y is so. Of course Not Y means NOT BEING IN PRISON, and Not X means NOT NOT LEARNING ENOUGH, which of course means LEARNING ENOUGH.. So If NOT IN PRISON, then it follows that you are LEARNING ENOUGH! Yes, I see how it works", finished off Alice with a flourish.

"There is something else we can say about our set", said Okto, trying to lead them on, "In [ban, bat, bin, fan, fat, fin] every word is either an a-word or an n-word"

"Or both", chimed in Kara.

"Let us agree that when we say either-or, we allow the possibility of both being the case", suggested Okto. "This is called an inclusive either-or"

"So when we use the ADD move, we create an either-or set", added Kara, "What kind of set do we get when we use the KEEP move ?"

"You would get the set [bat, ban] by doing KEEP a-words then KEEP b-words", said Okto, "In this set all words are BOTH a-words AND b-words. So instead of an either-or set, we have a both-and set. Also, the six other words form the set [bin, bit, fan, fat, fin, fit }, clearly an either-or set! Every word in this second set is EITHER an i-word OR an f-word. The i-words are those that are NOT a-words, the f-words are those that are NOT b-words."

"This is like the trick we used for solving the opposite of a problem!", exclaimed Alice. "That ancient person, De Morgan, did you call him? He must have thought about it by looking at either-or sets and both-and sets!"

"You are most likely right on that", said Okto, "If we call the i-words NOT a-words and the f-words NOT b-words, we can see better what is happening. We can say that

NOT BOTH a-word AND b-word comes to saying EITHER NOT a-word OR NOT b-word

And this is what we call the DE MORGAN RULE, in honor of Mr. De Morgan who was the first to think of it."

"This is getting pretty deep stuff", said Alice, "I didn't know we could get to so much brain twisting starting with a few easy games!"

At this point all the children decided that they had had enough brain twisting, but they also thanked Okto for the maths lesson, which they all declared had been highly enjoyable and hopefully useful in helping them with their thinking. The decided to go to the pool and to play one of the eightsome water polo games, the rules for all of which they had already learned to perfection.

