Bruce and Alice learn some Algebra
by Zoltan P. Dienes

It soon became the understood thing that Bruce, Alice, Unta, Ata and Alo went to school with the other local children. They soon got used to the base eight system of handling numbers. But there was another surprise awaiting them. One morning the mathematics teacher greeted them with the announcement:

"Today we are going to learn some algebra. You will need those trays, plastic cups and small pebbles that you see on the shelves", he said, pointing to the material neatly stacked on several shelves "Arrange yourselves in groups of four round the tables, with at least three trays per table. You will need a lot of cups, and even more pebbles"

"I thought algebra was to do with equations", whispered Bruce to Alice, as they were trying to organize themselves.

"What is troubling you?", asked the teacher, who heard Bruce's remarks.

"I have never heard of algebra being done with cups and trays and pebbles", admitted Bruce in a somewhat embarrassed tone, "When my friends at home do algebra, they just use paper and pencil, and talk about things called equations and factors!"

"Do you know what equations and factors are?", asked the teacher.

"Actually, I don't", replied Bruce, "We have not got as far as that in our mathematics. But I shall be excited to see how we can learn about those things, using your materials!"

"Good", said the teacher, "I am sure you will enjoy our methods. You can use pencil and paper as well, but mostly you will need to use your brain!"

"Here is the first exercise", said the teacher, "Choose a number, preferably less than 12, but in case don't choose a very big one, or you will run out of cups and pebbles!. Each table can choose a different number. Then put this number of cups on one of your trays. Then put your chosen number of pebbles in each cup on the tray. "

The teacher waited for the children to complete this part of setting up the problem. Then he went on:

"Now take two cups, put them by the tray but not on it. Put your chosen number of pebbles into each of these two cups. Finally put just one pebble next to the two additional cups"

When all the groups had finished carrying out these instructions, the teacher told them what the problem was which they had to solve, like this:

"Now rearrange all the pebbles you have used, those on the tray and those off the tray, so that all the pebbles are in cups, and every cup has the same number of pebbles in it. Then note and write down the following:

How many more or less cups than your chosen number are there in your new arrangement?

How many more or less pebbles than your chosen number are there in each cup?"

The children took quite a long time to arrange their pebbles, some did not realize that the single pebble at the end also had to be included. When they had all finished and noted their answers in their exercise books, the teacher went on to say:

"Now each table must choose another number and do the whole thing again. "

They took a lot less time doing the task the second time. Alice put her hand up, as she had something to tell the teacher.

"What is it, Alice? Have you seen something you did not expect?", asked the teacher.

"Yes", replied Alice, "Bruce and I got the same two numbers both times! Each time we had one more cup in the second arrangement than our chosen number, and there was always just one more pebble in each cup when we had rearranged the pebbles!"

"Are there any other tables where you got these exact answers?", asked the teacher.

About half the class obtained one more for the cups as well as one more for the pebbles in each cup for the rearrangement.

"I wonder if those groups that did not get the one more answers could try again and see if they could make an arrangement with one more cup being used than the number they have chosen?", suggested the teacher.

It did not take long for all the tables to have also ended up with the same two answers.

"This is very strange!" suggested Ata, "I wonder if it always works out like that and if it does, I wonder why it does?"

"I think I know", said Alo, "There are just as many cups on the tray as the number of pebbles in one of the cups that is by the tray. So you could take all the pebbles out of one of these extra cups, and put one into each of the cups on the tray. Then you could put the final extra pebble into the second extra cup. Now there will be one more cup filled with pebbles than the chosen number, and there will be one more pebble in each cup than the chosen number"

"I see", chimed in Bruce, "and what you have described can be done with ANY chosen number, as you never said what that chosen number had to be!"

The teacher was very pleased with the discussion that had taken place, it saved him having to explain how this one more thing was bound to happen every time! He said:

"You have understood very well how all this happens every time! I am very proud of you that you could work it out without my having to tell you. So I can now give you the following homework:

Start with a trayful of cups as before, so that there are as many cups on the tray as the number of pebbles in each cup. Instead of putting two more cups out by the tray, now put four more cups by the tray, and fill these with pebbles as you have filled the ones on the tray. Then place four pebbles by the extra cupfuls. Now try to rearrange the pebbles in the cups, so that every cup shall have the same number of pebbles in it. Note the difference between the number of cups used in the final arrangement and the number of cups on the tray, as well as the difference between the number of pebbles in each cup in the final arrangement and the original number. Carry out the task with several numbers, until you get the same differences every time!"

After the lesson the children went out to play some of the tennis games they had learned the previous week. On the way to the courts Bruce said to Alice:

"My friend at home always uses X's and Y's when he does algebra. I wonder how these letters have anything to do with what we did with the cups and the pebbles!"

"Maybe your friend uses X as a very short word for the number of cups we put on the tray?", suggested Alice.

"No, that can't be", replied Bruce, "They never use trays and cups in their algebra lessons. They only use X's!.

"But look", went on Alice, "If X is the number of cups on the tray, then X times X is the number of pebbles on the tray. Then there are some more pebbles off the tray. These would come to 2 times X for the two extra cups and 1 more. So the total number of cups we use is

X times X   plus   2 times X   plus   1   pebbles

Wouldn't that be right?"

At this point they reached the tennis courts, and there was no time to think about Alice's idea of how the home algebra was like the Ruritanian algebra. Alice did not play very well, and she lost every game, as she could not help thinking of X's and Y's which were floating about in her head, floating around many trays, cups and pebbles in her mind's eyes. So she rarely hit the ball, and eventually Bruce suggested that they should get out of the game as they were spoiling it for the others. They did this and then sat down on a bench under the shade of a small papaya tree.

"Have you thought of how you would solve the homework problem?", asked Alice.

"I think all you have to do is to take two of the four extra cups, empty each one into the cups on the tray, putting one pebble into each cup on the tray out of the first extra cup, then putting one pebble into each cup on the tray out of the second extra cup", replied Bruce, "Then you take two of the extra four pebbles and put them into the third extra cup, then put the other two extra pebbles into the fourth extra cup. There will be two more pebbles in each cup than there were before, and we shall have used two more cups than the number we first put on the tray. So the answer is two more cups than on the tray, and two more pebbles per cup than before!"

"That really sounds good!", exclaimed Alice "But I now wonder how you would have to set up the problem so that we would finish with three more cups in the end than the number on the tray, and three more pebbles per cup than there were at the beginning!"

"You want too much!, said Bruce "Isn't it enough that I have done your homework for you?"

But Alice's brain would not stop going round. Finally, after some thought, she said to Bruce:

"I think we would need six extra cups, as from the first three extra cups we can increase the number of pebbles in a cup on the tray by three, and then how many extra pebbles would we need to fill the other three extra cups", Alice was really talking to herself, but Bruce heard, and could not help replying:

"You will need nine pebbles, so that you can have three for each of the fourth, fifth and sixth cups, the ones that were not emptied. So it seems that six more cups and nine more single pebbles (don't forget to call it 11 in class!) will give us the two three more answers."

"I have just thought of something!" said Alice. "Is it so that the number of extra cups off the tray is always going to be twice the number by which we want to increase our chosen number?"

"And the number by which we want to increase our chosen number for the end arrangement, when multiplied by itself, will always give us the number of extra pebbles?" chimed in Bruce.

"The way we could find out would be if we tried to see how we would have to set up the problem so that we get four more cups used in the end than our chosen number", suggested Alice.

"Oh yes", agreed Bruce quickly, "Twice four is eight (or 10 in Eightland!), so we shall need eight extra cups off the tray. Four times four is sixteen (don't forget to call it 20 in class!), so we shall need sixteen extra pebbles. The first four extra cupfuls can be emptied into the cups on the tray, thereby increasing the content of each cup by four. Then there are sixteen pebbles left to share amongst the fifth, sixth, seventh and eighth extra cups, which would then make every cup having four more pebbles than they had before, and we are using four more cups than we originally put on the tray!

At the next mathematics lesson the teacher asked Alice for her homework.

"Here it is", said Alice, "I have drawn it with the chosen number being 6, and then 7"

This is what Alice's homework looked like, the letter p standing for "pebble"

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          Tray                                   Six extra cups

PPP  PPP  PPP  PPP  PPP  PPP      PPP  PPP  PPP  PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP      PPP  PPP  PPP  PPP  PPP  PPP

                                                    p p p
                                 single pebbles →   p p p
                                                    p p p
----------------------------------------------------------------------


PPP  PPP  PPP  PPP  PPP  PPP       PPP PPP PPP       P P P
PPP  PPP  PPP  PPP  PPP  PPP       PPP PPP PPP       P P P
PPP  PPP  PPP  PPP  PPP  PPP                         p p p

----------------------------------------------------------------------

PPP  PPP  PPP  PPP  PPP  PPP             PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP             PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP             PPP  PPP  PPP

----------------------------------------------------------------------:
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And here it is with the chosen number being 7:

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              TRAY                        CUPS                       1
PPP  PPP  PPP  PPP  PPP  PPP  PPP 1 PPP  PPP  PPP  PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP  PPP 1 PPP  PPP  PPP  PPP  PPP  PPP
P    P    P    P    P    P    P   1 P    P    P    P    P    P
                                  1

                                                  P P P
                              SINGLE PEBBLES      P P P
                                                  P P P

----------------------------------------------------------------------

PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP
PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP  PPP
 P    P    P    P    P    P    P    P    P    P
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"I can see you have been working on the problem, Alice", said the teacher, "Did you have much difficulty?"

"Oh no!", said Bruce and Alice together "We also solved the problem for four more, in fact for any number more that you want to end up with!", added Alice.

"Would you like to tell the class about it?", suggested the teacher.

"It is simple!", said Alice, "Suppose you want to end up with a certain number more cups than your chosen number, with exactly this number more pebbles than your chosen number in each cup. Let me call this number the difference number. Take two times this difference number, and this will be the number of extra cups you will have to fill and put by the side of the tray. Then you multiply the difference number by itself, and this will be the number of single pebbles you put by the extra cups. We checked it with the difference number being four, we have not yet checked it for 5 or 6", concluded Alice.

"This is where we can use the X's and Y's that your brother was talking about the other day", said the teacher.

"I did wonder about that", said Alice, "I suppose our chosen number could have the short name X, and the difference number we decide on could have the short name Y"

"You have hit the nail on the head!", said the teacher smiling, "Let us see if we can write down what you have told me using these short names."

He wrote the following on the chalk board:

X times X + 2 times X times Y + Y times Y is what we start with

Then X + Y multiplied by X + Y is what we end up with.

So whatever numbers we choose for X and for Y, it seems that the two must always come to the same number.

"Isn't there an even shorter way of writing X times X ?", asked Alo.

"Yes, there is", replied the teacher "It is even usual to leave out the word or the sign for times, whenever it does not lead to any confusion"

"Isn't a number multiplied by itself called the square of that number?", asked Ata.

"Yes, it is", replied the teacher, and you put a little 2 at the upper right side of the number that you are multiplying by itself"

"Is it like this?", asked Unta, running to the chalkboard and writing quickly with a small piece of chalk and saying aloud:

" $3 \times 3 = 3^2 = 11$, and $4 \times 4 = 4^2 = 20$ "

"That is how it is usually done", replied the teacher, "except Bruce and Alice would write their numbers in the base of ten which we call one two and write 12"

"I would like to write our discovery in the short way!", said Bruce, "May I?"

"All right, Bruce", said the teacher, "Come to the front and write what I wrote, but using the small 2 on the upper right corner for multiplying a number by itself. Also leave out the multiplication signs, you might confuse them with the letter X!"

Bruce went to the blackboard and wrote the following on it:

$$X^2 + 2XY + Y^2 = (X + Y)^2$$

"Why did you write those brackets?", Alice ventured to ask.

"I thought we should make sure that we do not give the impression that we want to add to X the number Y multiplied by itself, which then would have been written as

$$X + Y^2$$

without any brackets. First I wanted to put the $X + Y$ inside a circle, to make sure that we multiplied $X + Y$ by itself and not just the Y. Then I remembered that we could use brackets for this!"

" I have another question", asked Unta "How would you set up the problem so that the difference number for the number of cups is not the same as the difference number for the number of pebbles in each cup?"

"Good question", said the teacher, "How about studying this question for your homework for tomorrow?"

All the children thought this was a good idea, since the bell was just about to go to signal the end of this interesting mathematics lesson.

As they were leaving the classroom, Ata asked Bruce

"Would you know how to find such a problem? "

"Easy!", said Bruce, "Just put three extra full cups by the tray and put two pebbles by the extra cups!"

"I see!", said Ata, "You empty the contents of two of the extra cups into the cups on the tray, and put the two single pebbles into the third extra cup. Every cup will have two more pebbles in it than before, and the number of cups used will now be just one more than the number of cups we put on the tray. So the cup difference number is 1 and the pebble difference number is 2!"

"There you are!", said Bruce, "You have already done tomorrow's homework!"

"With a bit of help from you, Bruce, thank you very much!" said Ata as she walked off with her sister Unta

"When Bruce and Alice went back to their quarters, Alice said to Bruce:

"Let's write your example down in the teacher's short way!"

"All right", said Bruce, taking a pad of paper and a pencil out of the drawer, "Let X mean the number of cups we put on the tray (as well as the number of pebbles we put in each cup, of course), Then X^2 is the number of pebbles on the tray. Also 3 X is the number of pebbles in the three extra cups (short for 3 times our chosen number). So the total number of pebbles we are dealing with is

$$X^2 + 3X + 2$$

At the end we use (X + 1) cups, with (X + 2) pebbles in each cup, so we have

$$X^2 + 3X + 2 = (X + 2)(X + 1)$$

I think that is how I have seen it written in algebra books. Now I can see what it all means!" concluded Bruce.

"Could you ever end up with less cups than what we start with?", inquired Alice.

"I think you would have to take some pebbles away, instead of adding all those cups full of pebbles!",. suggested Bruce. "Suppose we had just one tray, with as many cups on it as the number of pebbles in a cup and that somebody just stole one pebble. I wonder how you would fix it so that all the cups had the same number of pebbles in it?"

"Oh, that's easy!" said Alice, "empty all the pebbles from the cup from which one has been taken and put one each in the remaining untouched cups. There will just be enough pebbles to go round, and we shall end up with one less cup than at the start and one more pebble per cup than at the start"

"And this is how we could write it", concluded Bruce, writing this on a pad of paper:

$$X^2 - 1 = (X + 1)(X - 1)$$

At the next algebra lesson Bruce was determined to bring up the problem of getting less cups in the rearranged set of pebbles. When they had settled themselves in groups, Bruce put the following question to the teacher:

"How can you end up with less cups instead of ending up with more cups than we put on the tray?"

"Obviously by removing some pebbles", chimed in Alice, remembering their previous conversation on the subject, "the problem is how many pebbles to remove, and out of how many cups?"

"Let us see somebody telling us about last night's homework first", suggested the teacher", does anybody want to share his or her work with the class?"

Ata volunteered and showed them her work. Here it is:

(One tray full of cups) + (5 extra cups filled) + 6 single pebbles

will always go into 2 more cups than the number on the tray with three more pebbles per cup than before or else it will go into 3 more cups than the number on the tray with two more pebbles per cup than before.

They all tried this out with different starting numbers for the number of cups on the tray, and they all checked that it worked in both the ways in which Ata had written in her homework Bruce volunteered to write Ata's result on the board, using X's. He knew that X meant the initial number of cups on the tray, so he wrote:

$$X^2 + 5X + 6 = (X + 2)(X + 3)$$

Which was the "shorthand" way of writing what Ata had written in her exercise book.

Unta put her hand up. When the teacher asked what she wanted to say, Unta said:

"Why not try Ata's problem but instead of adding five cupfuls, take away 5 cupfuls? And we could leave the 6 single pebbles as Ata had them. I believe in an inquiry it is always better to change just one thing at a time. Then we are more likely to find out how the change was caused!"

"Very smart, Unta", said the teacher, "Let's get on and find out. Of course you will have to fill enough cups for the tray so that you can remove five of them!"

All the children got busy with their pebbles and cups. Soon Alice got very excited and wanted to say something to the teacher.

"What is it, Alice?", inquired the teacher.

"I have an idea how to do it without knowing how many cups we put out on the tray at the start!"

"Tell us about it", said the teacher encouragingly.

"First you make the tray, and place the six single pebbles by the tray", Alice started explaining, "Then you remove three pebbles from each cup. That is like removing three cupfuls! Then you take two of the remaining cups, put three of the single pebbles into one of them, and the other three into the other one. These two cups have now become full cups again. So we remove them. In this way we shall have removed five cupfuls. There are two less cupfuls on the tray now than there were before, and each cupful has three less pebbles in it than before! So the difference number for the cups is 2 less, the difference number for the pebbles is 3 less!"

"That is very good indeed", said the teacher admiringly, "Let us all carry out Alice's instructions with different numbers and check that it works all the time, although I can see that her reasoning should have convinced you that it will work!"

They all checked Alice's statement and they all were satisfied that it worked. Then Alo had another problem:

"Can you add or take away ANY NUMBER OF CUPS, and/or ANY NUMBER OF PEBBLES, so that the pebbles can be rearranged into cups all with the same number of pebbles in them?", asked Alo.

"The answer to your question is in the negative", replied the teacher, "Once you have decided on the cups you add or take away, then only a few possible numbers of single pebbles remain , sometimes only one number of single pebbles, that will make the problem possible. Or you can decide on the pebbles, then you have to be careful of how many cupfuls you add or take away!"

"But how do you work out how many", asked a lot of children at once.

"I am afraid the bell is about to ring, so that problem must wait until we meet again!", said the teacher decidedly, just as the bell began to sound and the lesson was over.

Do you know the secret of the second number?

All the children were getting excited about wanting to know the "secret", and some of them tried to work it out before the next mathematics lesson. Bruce and Alice discussed nothing else the whole evening, and then drew in Ata, Alo and Unta as well as some of the local children. Their method of inquiry was to solve a lot of problems and note how the extra cups and pebbles were to do with the difference numbers for the cups and for the pebblkes. They had solved dozens of problems, then sorted them out, plotting the difference numbers against the extra cup and the extra pebble numbers. They did not try any take away problems, as they thought that would complicate things too much. When they had solved the problem for more cups and more pebbles, they would go on to less cups and less pebbles!

Here is the list of the problems and the way in which they had recorded the solutions:

Number of extra cups (not on tray)	2	3	4	4	5	5	6	6	6	7	8	8	8	9
Number more cups in the end than on tray at start	1	2	2	3	4	3	5	4	3	3	1	2	3	2
Number more pebbles per cup than at the start	1	1	2	1	1	2	1	2	3	4	7	6	5	7
Number of pebbles not in cups at the start	1	2	4	3	4	6	5	8	9	12	7	12	15	14

Bruce and Alice went to bed with these numbers floating about in their heads, but they were too tired to work out how they made an overall pattern. They thought there must be a pattern, as the rule of construction was always the same, so something the same should come out at the end of the construction!

Alice woke up after midnight, put her light on and had another look at the numbers. Sge started talking to herself:

"How stupid can you get? If you add the two difference numbers, you always get the number of extra cups that have to be placed by the tray! But if you multiply them, you get the number of single pebbles that must be placed next to the extra cups!" So for example if I am going to place nine extra cupfuls by the tray, then the difference numbers can be pairs such as these (1, 8), (2, 7), (3, 6), (4, 5) and so on, and so with these nine extra cupfuls I could have the following number of extra pebbles: 8, 14, 18, 20 and so on! All of these will work!"

She could not wait, so she walked along the passage and knocked on the door of the room where Bruce and Alo were sleeping.

"What is it? Is there an earthquake, or something?", came a sleepy voice from inside.

"No!", said Alice, "I have solved the problem, and I must tell you!"

"Come in, then", said Alo and Bruce together, but not quite awake yet.

Alice hurried into their room and quickly showed them her discovery. Alo and Bruce were duly impressed, but Bruce had the cheek to say:

"Now you can solve it for cups and pebbles being taken away!"

"Now that's an appreciative brother!", said Alo, "You should be proud of your sister, and happy that she wants to share things with you. I always share things with my sisters!"

"All right", said Bruce "I know you are getting good at these things, Alice, I am probably just a tiny bit jealous, so please forgive me!"

"Done!", replied Alice at once, "But I think we all need a sleep, so we shall wait for the rest of the problem till tomorrow! Good night Bruce! Goodnight Alo!", and she hurried out of the room and retired to her room and was soon fast asleep.

The next mathematics lesson was taken up mostly by Bruce, Alice and Alo explaining their discovery of the "secret", which was duly applauded by the rest of the class. But Alice was not too shy to say to the class:

"This is only a part of the problem solved. We need to know what happens when we take things away instead of adding things. Has anyone any ideas how our rule should be changed for such cases?"

"It seems easy when we take away cups but add single pebbles", suggested Unta "We still have to add the difference numbers, but then remove this number of cups. The number of single pebbles we still get by multiplying the difference numbers. This works when you remove five cupfuls and add six single pebbles, as we saw the other day"

"So if we remove six cupfuls and add 1 1 (nine in English) single pebbles, we shall end up with three less cups than the number on the tray and three less pebbles in each cup than at the start", interjected one of the local children, who did not want to be left out..

"But what happens if we take away single pebbles and add cupfuls?" asked Alice.

"Try adding one cupful to the trayful, and then steal six pebbles. How would you rearrange the pebbles so that every cup had the same number of pebbles in it?", asked the teacher.

Alo was ready with the solution. He said:

"That isn't very hard. Steal two of the pebbles from the extra cupful and two from one of the cupfuls on the tray and another two from another cupful on the tray. Then get two empty cups. Put one pebble into this empty cup from each of the untouched cups on the tray, and do the same with the other empty cup. Each cup will now be missing two pebbles, and we are using three more cups than the number we had on the tray at the start. So the cup difference number is 3 more, the pebble difference number is 2 less. Then 3 more and 2 less together are 1 more, which is the one extra cup we placed at the start. We also have 2 times 3 being equal to 6, but instead of adding, we take away the pebbles. What I am not so sure of is how we know when we have to add pebbles and when we have to remove them", finished Alo, quite out of breath.

Alice made the following suggestion:

"When the difference numbers are both more numbers, or when they are both less numbers, it seems we have to add the single pebbles. Perhaps when one of them is a more number and the other is a less number that we have to remove as many pebbles as the product of these two difference numbers. It's just a thought!", said Alice modestly, "It might not always work out, but maybe we should try!"

"It certainly works on our example when we stole six pebbles", added one of the children.

"Let us do one when we remove cupfuls but add single pebbles", suggested Alo.

"We have already done one like that", objected several children, "We ought to try one when we remove cupfuls as well as single pebbles. In any case at the start we do not know the difference numbers, so how can we tell if one will be a more number and the other a less number?" added Ata.

"What about stealing a whole cupful as well as six single pebbles?", suggested Bruce, who was already trying to solve it in his head.

"Let me try!", said Ata, "I think I can do it like this: After I have stolen the whole cupful, I will steal three pebbles from one cupful on the tray, then another three pebbles from another cupful on the tray. Now I shall get three empty cups. I shall put a pebble from each untouched cup on the tray into the first empty cup. I shall do likewise with the second empty cup, then I shall do the same again with the third empty cup. Since there were three less untouched cups on the tray after all the stealing than before, that will be the number of pebbles in each of the three new cups I shall have used. So there will be two more cups in use than the original number of cups on the tray (as don't forget we stole one, but put three new ones there!), and each cup will now have three pebbles less in it than at the start So the difference number for the cups is two more, the difference number for the pebbles is three less. Luckily, one is a more number and the other is a less number, so by Alice's rule we must steal 2 times 3 pebbles, which is just what we did! "

"We could have stolen one cupful as well as 1 4 single pebbles (12 in English)", suggested Unta "the difference numbers being 4 less and 3 more. I think we have really solved the problem of the secret!", concluded Unta. "Once we know how many cupfuls we have to add or take away, we can easily set a problem that can be solved by saying how many single pebbles must be either added or taken away. And we can do it the other way round. Given the number of single pebbles, and knowing whether they are to be added or removed, we can provide a suitable number of cupfuls to be either added or removed!"